Some General Aspects of Coset Models and Topological Kazama-Suzuki Models

Matthias Blau[1]

Laboratoire de Physique Théorique ENSLAPP[2]
ENSLyon, 46 Allée d'Italie,
F-69364 Lyon CEDEX 07, France

Faheem Hussain[3] and **George Thompson**[4]
ICTP
P.O. Box 586
34014 Trieste, Italy

We study global aspects of $N = 2$ Kazama-Suzuki coset models by investigating topological G/H Kazama-Suzuki models in a Lagrangian framework based on gauged Wess-Zumino-Witten models. To that end, we first generalize Witten's analysis of the holomorphic factorization of bosonic G/H models to models with $N = 1$ and $N = 2$ supersymmetry, in the course of which we also find some new anomaly-free and supersymmetric models based on non-diagonal embeddings of the gauge group. We then explain the basic properties (action, symmetries, metric independence, ...) of the topologically twisted G/H Kazama-Suzuki models. As non-trivial gauge bundles unavoidably occur, we explain how all of the above generalizes to that case.

We employ the path integral methods of localization and abelianization (shown to be valid also for non-trivial bundles) to establish that the twisted G/H models can be localized to bosonic H/H models (with certain quantum corrections), and can hence be reduced to an Abelian bosonic T/T model, T a maximal torus of H. We also present the action and the symmetries of the coupling of these models to topological gravity. We determine the bosonic observables for all the models based on classical flag manifolds and the bosonic observables and their fermionic descendants for models based on complex Grassmannians. These results will be used in subsequent publications to calculate explicitly the chiral primary ring of Kazama-Suzuki models. IC/95/340 hep-th/9510187 ENSLAPP-L-556/95

---

[1]e-mail: mblau@enslapp.ens-lyon.fr; supported by EC Human Capital and Mobility Grant ERB-CHB-GCT-93-0252

[2]URA 14-36 du CNRS, associée à l'E.N.S. de Lyon, et à l'Université de Savoie

[3]e-mail: hussainf@ictp.trieste.it

[4]e-mail: thompson@ictp.trieste.it

Contents

1 Introduction

In this article we investigate some global aspects of Wess-Zumino-Witten (WZW) coset models in general, and of $N = 2$ Kazama-Suzuki models [1] in particular. WZW models [2, 3] are the basic building blocks of (rational) conformal field theories and are, in principle, exactly solvable due to their large symmetry algebras. As such they form one of the cornerstones of our present understanding of string theory. On the other hand, world-sheet theories with $N = 2$ superconformal symmetry not only display a very rich algebraic structure [4] but are also phenomenologically attractive as they lead to $N = 1$ space-time supersymmetry.

Of particular interest, therefore, is a class of world-sheet theories, known as Kazama-Suzuki models, combining these two features. This large class of $N = 2$ theories was discovered by Kazama and Suzuki [1] by determining under which conditions the $N = 1$ super-GKO coset construction [5] for a G/H coset conformal field theory actually gives rise to an extended $N = 2$ superconformal symmetry. For a complete classification, see [6].

Subsequent investigations of these models, using the powerful operator for-

malism of (super-)conformal field theories [4, 7, 8, 9, 10, 11], have brought to light the rich and beautiful structure underlying these models, arising from the interplay between the geometry and topology of coset spaces on the one hand and the $N = 2$ algebra on the other. Among the most striking results is the emergence of global (topological) structures (like the relation of the chiral primary ring [4] to cohomology rings) from these essentially local considerations.

In such a situation it is desirable to understand these results in a different way, perhaps from an inherently more global point of view which might be capable of shedding some light on the raisons d'être of these features of a field theory. Now, certain global aspects of (two-dimensional) field theories are frequently easier to extract from an action-based path integral formulation of the theory. The loss of flexibility in working with a fixed action is then compensated by the ability to employ other powerful methods of (gauge) field theories like the path integral formalism which are quite complementary to the standard conformal field theory techniques.

Coset models were first considered by Halpern et al [12]. The Lagrangian approach to coset WZW models was pioneered by Gawedzki and Kupiainen [13] and Karabali and Schnitzer [14] in the bosonic case and by Schnitzer [15] for the $N = 1$ models. Based on this, a Lagrangian realization of Kazama-Suzuki models was then provided in [16, 17]. There it was shown that, under the conditions for $N = 2$ superconformal symmetry determined in [1], also the Lagrangian realization of the $N = 1$ coset models as supersymmetric gauged WZW models [15] actually possesses the expected $N = 2$ superconformal symmetry.

Some aspects of this Lagrangian realization were investigated further by Nakatsu and Sugawara [18] and Henningson [19, 20]. In particular, in [18] the relation between the gauge-theoretic and the more standard conformal field theory realization of these models was clarified by showing how systematic use of the 'complex gauge' trick [13, 14] leads to a free-field representation of Kazama-Suzuki models.

Our starting point for this and the subsequent paper [21] is the work of Witten [16] who analyzed in detail the topologically twisted $G/H = CP(1)$ models (a.k.a. $N = 2$ minimal models) and their coupling to topological gravity. In fact, our main aim in this paper, which deals with the more formal and general aspects of Kazama-Suzuki models, will be to (partially) generalize the analysis of Witten to arbitrary G/H Kazama-Suzuki models.

The results will then be used in [21] and subsequent publications to perform explicit calculations in these models.

In particular, we will establish here that a topological G/H Kazama-Suzuki model can be reduced (localized) to a bosonic Abelian topological field theory, and we present the coupling of an arbitrary G/H model to topological gravity. The former simplifies significantly the task of performing calculations in these models. The latter provides us with a Lagrangian realization for a large class of topological matter-gravity systems and hence topological string theories and a corresponding generalization of the intersection theory studied in [16].

While the main subject of this paper are the twisted $N = 2$ models, it is natural to proceed to them via the original bosonic cosets and then their supersymmetric extensions. In order to gain a good understanding of the topological Kazama-Suzuki models, which are not quite topological field theories of the (well understood) cohomological kind, we find it useful to start with the approach to (bosonic) coset models suggested by Witten in [3], based on wave functions constructed from anomalously gauged WZW models. We first generalize this analysis to models with $N = 1$ and $N = 2$ supersymmetry. While this generalization is quite straightforward, at least formally[1] and in principle, it sheds some light on some of the issues we will discuss in the more specific context of topological Kazama-Suzuki models later on (the supersymmetries, the metric (in-)dependence, the coupling to topological gravity). As a by-product it also provides us with plenty of new anomaly-free supersymmetric (and topological) models arising from non-diagonal embeddings of the gauge group. We will, however, postpone a more detailed discussion of these new models and just sketch their construction here.

The other ingredient in our analysis of the topological Kazama-Suzuki models are various localization techniques for functional integrals (see [22] for a review). A combination of the localization techniques used e.g. in [16] and [23] with the method of diagonalization introduced in [24] (and analysed further in [25, 26] - see also [27] for very nice recent applications of this technique) will allow us to simplify the original theory to the extent that explicit calculation of correlation functions becomes feasible. In fact, we will be able to reduce the original non-Abelian, non-linear and supersymmetric

[1] Our discussion in that section will be at a level where we do not regularize explicitly quadratic operators - the required quantum corrections are, however, known to lead to the familiar shifts in the level k.

topological G/H Kazama-Suzuki models to a bosonic Abelian T_H/T_H model (T_H a maximal torus of H) - with certain quantum corrections which in this section we will take care to keep track of.

As the partition function of the topological Kazama-Suzuki models generically vanishes because of ghost number anomalies, to make the above result meaningful and useful we also determine the (classical) observables in a number of models (complex Grassmannians, flag manifolds G/T_G of classical groups G). In all these models we find that the competing demands of gauge invariance and supersymmetry can be met and yield r (the rank of G) independent generators of the classical algebra of observables.

In [21], we will then use these result to get a fairly explicit expression for correlators in topological Kazama-Suzuki models based on complex Grassmannians, and to establish the relation (mentioned above) between the ring of topological observables and the classical cohomology ring of the corresponding Grassmannian. What is interesting from the present point of view of gauged WZW models is that, while one is gauging the adjoint action of H on G, the cohomology one finds is not that of the (rather singular) space $G/\mathrm{Ad}H$ but rather that of the right (or left) coset G/H. We still have no completely satisfactory explanation for how this comes about, save in a particular topological sector in which essentially only the fermionic part of the theory plays a role which is more obviously related to the ordinary coset G/H (or, rather, its tangent bundle) [21].

We should warn the reader that through most of the paper we will make use of the usual gauged Wess-Zumino action. That is, we will treat all the fields as appropriately valued maps. This does not really suffice when one is working with non-trivial bundles, as we are forced to, for then the fields are really sections. However, we proceed as if the fields are maps and return, at the end, to the subtleties involved in defining the gauged Wess-Zumino-Witten term for non-trivial bundles. We will see that our conclusions based, as they were, on the naive considerations at the start of the paper are indeed correct.

More systematically, this paper is organized as follows. In section 2 we review the reults of [3] on the holomorphic factorization of the partition function of bosonic coset models and generalize them in two ways, extending the analysis to correlation functions and $N = 1$ supersymmetric coset models. We also sketch a construction of some new anomaly free supersymmetric models arising in this way.

In section 3, we deal with the models of primary interest in this paper, namely $N = 2$ coset models and their topological partners. We recall the Lagrangian realization of the $N = 2$ and topological Kazama-Suzuki models and their basic algebraic and symmetry properties. We also establish the metric independence and holomorphic factorization of the twisted models from the wave function point of view.

In section 4, we discuss the localization of the topological G/H Kazama-Suzuki model to a bosonic H/H model (generalizing the corresponding result for $SU(2)/U(1)$ in [16], albeit using a slightly different method), and the subsequent abelianization to a T_H/T_H model (following [24]).

In section 5, we first of all give a direct proof that the model we are dealing with is a topological field theory by showing that the energy-momentum tensor of the twisted Kazama-Suzuki model is BRST exact modulo the equations of motion of the gauge field. We then present the coupling of the topological Kazama-Suzuki model to topological gravity. This coupling turns out to be the more or less obvious generalization of Witten's result for $SU(2)/U(1)$ [16] in the case that G/H is hermitian symmetric. In general, however, one finds that certain additional terms are also present in the Lagrangian.

In section 6 we analyze the bosonic observables in the classes of models mentioned above, as well as their fermionic descendants (pointing out the complications that arise when applying the usual descent procedure, familiar from cohomological field theories, in the somewhat different present setting).

Finally, in sections 7 and 8 we return to the issue of what the definition of the theory should be when the bundles in question are non-trivial. The definition that we take is adapted from that of [28] and [29]. The main technical result that we obtain is an appropriate generalization of the Gawedzki-Hori definition to anomously gauged WZW models, i.e. wavefunctions. On the basis of that result it is then practically guaranteed that the analysis performed in sections 2 through 6 remain valid in the general setting and we verify this explicitly. Nevertheless, as pointed out by Hori [29], taking account of the non-trivial topological sectors is important if one is to resolve the 'fixed point' problem [30] that arises in the context of conformal field theory.

As should be clear from the above description of what we do and what we do not do in this paper and in [21], this is but a first step towards a full understanding of the global properties of Kazama-Suzuki models.

For example, one would like to have a better *a priori* understanding from the present point of view of what it is that one is calculating in these models (namely, cohomology rings of homogeneous spaces and generalizations thereof [4, 10, 21]).

Clearly, also the topological matter systems coupled to topological gravity we construct in section 5 are worthy of further study, in particular, their interpretation in terms of intersection theory on (a suitable cover of) the moduli space of curves (as in [16]). Also the description of these models in terms of integrable hierarchies associated with G/H (presumably related to the hierarchies studied in [31]) needs to be elucidated.

2 Factorization of Bosonic and $N = 1$ Coset Models

We begin with a brief review of the salient features of gauged Wess-Zumino-Witten (WZW) models. Sections 2.1, 2.2 and 2.4 are essentially a review of the pertinent results of [3] while the other parts of this section contain generalizations thereof (primarily to supersymmetric models).

The action of the WZW model at level k for a compact semi-simple group G is $kS(g)$ where

$$S(g) = -\frac{1}{8\pi}\int_{\Sigma} g^{-1}dg * g^{-1}dg - i\Gamma(g), \tag{2.1}$$

and

$$\Gamma(g) = \frac{1}{12\pi}\int_{M}(g^{-1}dg)^3 \tag{2.2}$$

with $\partial M = \Sigma$ and g a map form the two-dimensional closed surface Σ to the group G (g also denotes its extension to M). Here and in the following a trace Tr (Killing-Cartan form of G) will always be understood in integrals of Lie algebra valued forms. We also assume that Tr has been normalized in such a way that the quantum theory is well-defined (independent of the extension of g to M) for integer k.

The action $S(g)$ has a global $G_L \times G_R$ invariance, $g \to h^{-1}gl$. It is well known, however, that one can not gauge any subgroup of $G_L \times G_R$, due to anomalies. The condition on the subgroup F such that it be anomaly free is as follows. Let Tr_L and Tr_R be the traces on $\mathrm{Lie}\, G_L$ and $\mathrm{Lie}\, G_R$. Then, if for all $t, t' \in \mathrm{Lie}\, F$

$$\mathrm{Tr}_L\, tt' - \mathrm{Tr}_R\, tt' = 0 \ , \tag{2.3}$$

the subgroup F is anomaly free. The standard example is when $F = H \times H$ (acting diagonally as $g \to h^{-1}gh$) and the resulting theory is known as *the* G/H model.

2.1 Gauged WZW Models

Witten [3] has shown that the WZW action gauged by anomalous subgroups is worthy of further study. One may use these non-gauge invariant models to establish the holomorphic factorization of the path integral of anomaly free gauged WZW models.

Let $F = H_L \times H_R \subseteq G_L \times G_R$ be a, possibly anomalous, subgroup. Denote the Lie algebra of $H_L \times H_R$ by $\mathfrak{h}_L \oplus \mathfrak{h}_R$. The field g that enters in the gauged model is correctly thought of as a section of a $H_L \times H_R$ bundle $X \to \Sigma$. Let (A, B) be a $(H_L \times H_R)$ connection on X.

The action we wish to consider is

$$\begin{aligned} S(A,B;g) &= -\frac{1}{8\pi}\int_\Sigma g^{-1}d_{\mathcal{A}}g * g^{-1}d_{\mathcal{A}}g - i\Gamma(g) \\ &\quad + \frac{i}{4\pi}\int_\Sigma \left(Adgg^{-1} + Bg^{-1}dg\right) + \frac{i}{4\pi}\int_\Sigma Bg^{-1}Ag \end{aligned} \tag{2.4}$$

where the covariant derivative is defined by

$$d_{\mathcal{A}}g = dg + Ag - gB\,. \tag{2.5}$$

The action can be usefully expressed as

$$\begin{aligned} S(A,B;g) &= S(g) + \frac{1}{4\pi}\int_\Sigma B(i+*)g^{-1}dg + \frac{1}{4\pi}\int_\Sigma A(i-*)dgg^{-1} \\ &\quad - \frac{1}{8\pi}\int_\Sigma (B*B + A*A) + \frac{1}{4\pi}\int_\Sigma B(i+*)g^{-1}Ag. \end{aligned} \tag{2.6}$$

In terms of local co-ordinates, this also fixes our conventions, this reads,

$$\begin{aligned} S(A,B;g) &= S(g) + \frac{1}{2\pi}\int_\Sigma d^2z\,\mathrm{Tr}\,B_{\bar z}g^{-1}\partial_z g - \frac{1}{2\pi}\int_\Sigma d^2z\,\mathrm{Tr}\,A_z\partial_{\bar z}g.g^{-1} \\ &\quad + \frac{1}{2\pi}\int_\Sigma \mathrm{Tr}\,A_z g B_{\bar z}g^{-1} - \frac{1}{4\pi}\int_\Sigma d^2z\,\mathrm{Tr}\,(A_zA_{\bar z} + B_zB_{\bar z}) \end{aligned} \tag{2.7}$$

As we noted before, the action (2.6) is not gauge invariant in general, but it is the next best thing. Under the transformations,

$$A \to A^h = h^{-1}Ah + h^{-1}dh, \quad B \to B^l = l^{-1}Bl + l^{-1}dl, \quad g \to h^{-1}gl \tag{2.8}$$

we find that

$$S(A,B;g) \to S(A,B;g) + \frac{i}{4\pi}\int_\Sigma \left(Bdl.l^{-1} - Adh.h^{-1}\right) - i\Gamma(l) + i\Gamma(h). \quad (2.9)$$

The attractive feature here is that the variation of the action depends neither on the metric nor on the group element g.

There are various ways to construct a gauge invariant action, that is to satisfy (2.3). A simple choice is the standard diagonal embedding, $H_L \times H_R = H \times H$ for $H \subseteq G$, with $(A,B) = (A,A)$. Somewhat more exotic is the case where, if $H = H' \times U(1)$, we can have a left connection $A' + a$ and a right connection $B' + b$ such that

$$A' = B' \quad , \qquad a = -b \,, \qquad (2.10)$$

which corresponds to the so called axial gauging of the coset model.

2.2 Holomorphic Factorization of Bosonic Coset Models

While the action (2.6) is not invariant under gauge transformations, we may, nevertheless, use it to create a wavefunction which transforms in a well prescribed way. Let us consider the wavefunction

$$\Psi(A,B) = \int Dg\, e^{-kS(A,B;g)} \quad . \qquad (2.11)$$

Under a gauge transformation, Ψ transforms as

$$\begin{aligned} \Psi(A^h, B^l) &= \mathrm{e}^{ik\Phi(A;h) - ik\Phi(B;l)}\Psi(A,B) \\ &= \mathrm{e}^{-\frac{ik}{4\pi}\int_\Sigma \left(Bdl.l^{-1} - Adh.h^{-1}\right) + i\Gamma(l) - i\Gamma(h)}\Psi(A,B) \end{aligned} \qquad (2.12)$$

The phase factors $\Phi(A;h)$ and $\Phi(B,l)$ are cocycles. E.g. with $h' \in H_L$, $\Phi(A,h)$ satisfies

$$\Phi(A;hh') = \Phi(A^h;h') + \Phi(A;h) \quad , \qquad (2.13)$$

and likewise for B.

The $\Psi(A,B)$ are therefore correctly thought of as sections of a bundle, which we now determine.

Under an infinitesimal set of transformations

$$\delta A = d_A u, \quad \delta B = d_B v, \quad \delta g = -ug + gv, \tag{2.14}$$

with $u \in \mathfrak{h}_L$ and $v \in \mathfrak{h}_R$ the variation of the action is

$$\delta S(A, B; g) = \frac{i}{4\pi} \int_\Sigma \mathrm{Tr}\,(udA - vdB)\,, \tag{2.15}$$

so that Ψ satisfies

$$\begin{aligned} \left(D_\mu^A \frac{\delta}{\delta A_\mu} + \frac{ik}{4\pi} \epsilon^{\mu\nu} \partial_\mu A_\nu \right) \Psi(A, B) &= 0 \\ \left(D_\mu^B \frac{\delta}{\delta B_\mu} - \frac{ik}{4\pi} \epsilon^{\mu\nu} \partial_\mu B_\nu \right) \Psi(A, B) &= 0 \ , \end{aligned} \tag{2.16}$$

where the covariant derivatives are

$$D_\mu^A = \partial_\mu + [A_\mu, .] \quad D_\mu^B = \partial_\mu + [B_\mu, .] \ . \tag{2.17}$$

We also have the equations

$$\begin{aligned} \left(\frac{\delta}{\delta A_{\bar{z}}} - \frac{k}{4\pi} A_z \right) \Psi(A, B) &= 0 \\ \left(\frac{\delta}{\delta B_z} - \frac{k}{4\pi} B_{\bar{z}} \right) \Psi(A, B) &= 0. \end{aligned} \tag{2.18}$$

Introducing the operators

$$\begin{aligned} \frac{D}{DA_z} &= \frac{\delta}{\delta A_z} + \frac{k}{4\pi} A_{\bar{z}} \\ \frac{D}{DA_{\bar{z}}} &= \frac{\delta}{\delta A_{\bar{z}}} - \frac{k}{4\pi} A_z \\ \frac{D}{DB_z} &= \frac{\delta}{\delta B_z} - \frac{k}{4\pi} B_{\bar{z}} \\ \frac{D}{DB_{\bar{z}}} &= \frac{\delta}{\delta B_{\bar{z}}} + \frac{k}{4\pi} B_z \ , \end{aligned} \tag{2.19}$$

we may rewrite (2.18) in this new notation as

$$\begin{aligned} \frac{D}{DA_{\bar{z}}} \Psi(A, B) &= 0 \\ \frac{D}{DB_z} \Psi(A, B) &= 0, \end{aligned} \tag{2.20}$$

and (2.16) becomes

$$\begin{aligned}\left(D^A_\mu \frac{D}{DA_\mu} + \frac{ik}{4\pi}\epsilon^{\mu\nu} F(A)_{\mu\nu}\right)\Psi(A,B) &= 0 \\ \left(D^B_\mu \frac{D}{DB_\mu} - \frac{ik}{4\pi}\epsilon^{\mu\nu} F(B)_{\mu\nu}\right)\Psi(A,B) &= 0. \end{aligned}\tag{2.21}$$

Geometrically the situation can be described as follows [3]. Let $\mathcal{A}\times\mathcal{B}$ be the space of (A,B) connections on Σ, equipped with the symplectic two-form

$$\omega\left((a_1,b_1),(a_2,b_2)\right) = \frac{1}{2\pi}\int_\Sigma \mathrm{Tr}\, a_1\wedge a_2 - \frac{1}{2\pi}\int_\Sigma \mathrm{Tr}\, b_1\wedge b_2. \tag{2.22}$$

On $\mathcal{A}\times\mathcal{B}$ there is a prequantum line bundle $\mathcal{L} = \mathcal{L}_1^{\otimes k}\otimes\mathcal{L}_2^{\otimes(-k)}$, with $\mathcal{L}_1$ a line bundle on $\mathcal{A}$ and $\mathcal{L}_2$ a line bundle over $\mathcal{B}$. (2.20) and (2.21) taken together tell us that $\Psi(A,B)$ is an (equivariant) holomorphic section of the line bundle $\mathcal{L}$ over $\mathcal{A}\times\mathcal{B}$.

The reason for the interest in these wave functions is that for $B\in\mathfrak{g}$ the partition function of the G/H coset model is simply its norm,

$$Z_{G/H}(\Sigma) = |\Psi|^2. \tag{2.23}$$

In order to establish this we need to introduce the conjugate of the wave-function

$$\overline{\Psi(A,B)} = \int Dh\, e^{-kS(B,A;h)}. \tag{2.24}$$

with $h\in G$. We can compute the norm, by performing the Gaussian integral over B, with $B\in\mathfrak{g}$ and $h,l\in G$,

$$\begin{aligned}|\Psi|^2 &= \int DADBDlDh\, \mathrm{e}^{-kS(A,B;l) - kS(B,A;h)} \\ &= \int DADlDh\, \mathrm{e}^{-kS(A;lh)} \\ &= \int DADg\, \mathrm{e}^{-kS(A;g)}\ , \end{aligned}\tag{2.25}$$

which is the desired result. In passing from the first to the second line we have integrated over B and made use of the Polyakov-Wiegman identity and we have, as well, normalized the volume $\int Dh = 1$ throughout. Another way of arriving at this result, which bypasses the need to use the Polyakov-Wiegman identity, is to notice that as the gauge group associated with B is

now all of G one can gauge fix $l = 1$. The dependence on B is now essentially pure Gaussian with covariance 1.

Actually, one can overlap the wavefunctions even when B takes values in some subalgebra of $\mathfrak{g}$ and arrive at a gauge invariant theory. One may wonder what this theory is. It turns out to be an anomaly free gauged WZW model with action

$$S(A, B; g) + S(B, A; h). \tag{2.26}$$

This model will make sense, with $(g, h) \in G \times G'$ and $(A, B) \in (\mathfrak{h}, \mathfrak{h}')$ providing H and H' are subgroups of both G and G'. The coset model is then a $(G \times G')/(H_L \times H_R)$ theory where $H_L = (H, H')$ and $H_R = (H', H)$. When $H' = G' = G$ this reduces, as we have seen, to a G/H theory.

In this way we have reproduced a standard coset model and this is in itself not a particularly interesting observation. It gains interest, however, when one couples to fermions. We will briefly come back to this below.

2.3 Introduction of Observables

There is a class of observables in G/H coset models that are of prime importance. Let R_i be an irreducible representation of G. The observables in question are

$$\bigotimes_i \mathrm{Tr}_{R_i}(g(x_i)). \tag{2.27}$$

Let us denote the correlation function of these by

$$Z_{G/H}(\{R_i\}; \{x_i\}). \tag{2.28}$$

In order to establish holomorphic factorization for these correlation functions we will need to consider a more general class of wavefunctions. Let $D_{R_i}(g)$ be the matrix of the irreducible representation R_i of G acting on the finite dimensional representation space V_i. Define the wavefunction

$$\Psi(A, B, \{R_i\}, \{x_i\}) = \int Dg \mathrm{e}^{-kS(A,B;g)} \bigotimes_i D_{R_i}(g(x_i)). \tag{2.29}$$

Notice that these insertions are not gauge invariant. But as the action is also not invariant this situation is not completely problematic. If one considers pointed gauge transformations (i.e. those that do not act at the points

$\{x_i\})$ then the wavefunction transforms just as in the case without operator insertions considered previously. We also define a dual wavefunction by

$$\overline{\Psi(A,B,\{R_i\},\{x_i\})} = \int Dh \mathrm{e}^{-kS(B,A;h)} \bigotimes_i D_{R_i}(h(x_i)) \tag{2.30}$$

with the observables positioned at the same points and in the same representations as in (2.29).

Let $\mathrm{Tr}_{R_i}(hl)(x_i)$ denote $\mathrm{Tr}\, D_{R_i}(l(x_i))D_{R_i}(h(x_i))$. The derivation given in (2.25) can be followed through line by line to establish (again $B \in \mathfrak{g}$)

$$Z_{G/H}(\{R_i\},\{x_i\}) = \int DBDA \bigotimes_i \mathrm{Tr}_{R_i} |\Psi(A,B,\{R_i\},\{x_i\})|^2. \tag{2.31}$$

The new wavefunctions do not obey (2.21). Let $T^{a,b}_{L,R}$ be the generators of $\mathfrak{h}_{L,R}$. One finds instead of (2.21)

$$\begin{aligned}
& \left(D^A_\mu \frac{D}{DA_\mu} + \frac{ik}{4\pi}\epsilon^{\mu\nu}F(A)_{\mu\nu} \right) \Psi(A,B;\{R_i\},\{x_i\}) \\
= \quad & -\sum_i \delta(x-x_i) T^a_L D_{R_i}(T^a_L)\Psi(A,B;\{R_i\},\{x_i\}) \\
& \left(D^B_\mu \frac{D}{DB_\mu} - \frac{ik}{4\pi}\epsilon^{\mu\nu}F(B)_{\mu\nu} \right) \Psi(A,B;\{R_i\},\{x_i\}) \\
= \quad & \sum_i \delta(x-x_i)\Psi(A,B;\{R_i\},\{x_i\}) T^b_R D_{R_i}(T^b_R),
\end{aligned} \tag{2.32}$$

with the matrix multiplication understood to be on the appropriate factors.

2.4 Variation of the Complex Structure

So far, we have considered the actions for a fixed complex structure on Σ (entering via the Hodge star $*$ in the action). We will now look at what happens when one varies the complex structure (or conformal equivalence class of a metric ρ on Σ). Thus let $\mathcal{S}$ be the space of all conformal classes of metrics on Σ. The space of holomorphic and gauge invariant sections of $\mathcal{L}^{\otimes k}$ depends on the given metric ρ; we denote that vector space by W_ρ. As ρ varies over $\mathcal{S}$, W_ρ varies as the fibre of a vector bundle $\mathcal{W}$ over the space $\mathcal{S}$ of complex structures on Σ. Thus the wavefunctions we have constructed

can be thought of as sections of $\mathcal{W}$. Set

$$\delta^{(1,0)} = \int_\Sigma \delta\rho_{\bar{z}\bar{z}} \frac{\delta}{\delta\rho_{\bar{z}\bar{z}}}\,, \quad \delta^{(0,1)} = \int_\Sigma \delta\rho_{zz} \frac{\delta}{\delta\rho_{zz}}. \tag{2.33}$$

We notice that[2], when $B \in \mathfrak{g}$,

$$\nabla^{(1,0)}\Psi(A,B;\rho) \equiv \left[\delta^{(1,0)} + \frac{\pi}{2k}\int_\Sigma \delta\rho_{\bar{z}\bar{z}}\,\mathrm{Tr}\,\frac{D}{DB_{\bar{z}}}\frac{D}{DB_{\bar{z}}}\right]\Psi(A,B;\rho) = 0. \tag{2.34}$$

This defines an anti-holomorphic structure on $\mathcal{W}$. Similarly a holomorphic structure is defined by saying that a section is holomorphic if it is annihilated by

$$\nabla^{(0,1)} \equiv \delta^{(0,1)} - \frac{\pi}{2k}\int_\Sigma \delta\rho_{zz}\,\mathrm{Tr}\,\frac{D}{DA_z}\frac{D}{DA_z} \tag{2.35}$$

A straightforward exercise shows that if $A \in \mathfrak{g}$ then

$$\nabla^{(0,1)}\Psi(A,B;\rho) = 0. \tag{2.36}$$

The wavefunctions of the G/G models are therefore special in being holomorphic and anti-holomorphic. This implies that the norm-squared of the wave function, i.e. the partition function for the G/G theory, is constant as a function on $\mathcal{S}$ and hence metric independent, defining a topological field theory. It could have been that the partition function is metric independent without the stronger statement that it is the norm of a wavefunction that is both holomorphic and anti-holomorphic.

The operators $\nabla^{(1,0)}$ and $\nabla^{(0,1)}$ will figure prominently in the proof of metric independence of the topological models. Indeed, as we will see, for the topological $N = 2$ coset models with $\mathfrak{h} \subset \mathfrak{g}$ the wavefunctions indeed satisfy both (2.34) and (2.36) just as they do for the G/G model. One can therefore take this trait as a *definition* of the topological coset models.

Altogether (2.23) and (2.34) are the statement of holomorphic factorization of the G/H coset model. As $B \in \mathfrak{g}$, Ψ can be taken to be an anti-holomorphic section of $\mathcal{W}$, and consequently, if $e_i(B,\rho)$ form a holomorphic and orthogonal basis of $\mathcal{W}$ we can expand $\Psi(A,B;\rho)$ as

$$\Psi(A,B;\rho) = \sum_i \overline{e_i(B;\rho)}\Psi_i(A,\rho) \tag{2.37}$$

[2] In subsequent formulae one should regularize the currents that appear. One way to do this is to use a gauge invariant point splitting procedure [32]. The net effect is that one ought to replace k with $\overline{k} = k + c_H$.

where $\Psi_i(A,\rho)$ is anti-holomorphic as a 'function' on $\mathcal{S}$. The G/H partition function can, therefore, be expressed as

$$Z_{G/H}(\Sigma;\rho) = \int DA \ \sum_i |\Psi_i(A;\rho)|^2. \tag{2.38}$$

This establishes the holomorphic factorization not only for the standard diagonal embedding but for all the other anomaly free gauged models as well, *including* the axially gauged theories.

In the case of the G/G model, the wavefunction $\Psi(A,B;\rho)$ must also be an anti-holomorphic section of $\mathcal{W}$ so that we may write it as

$$\sum_{ij} \overline{e_i(B,\rho)} e_j(A,\rho) d^{ij} \tag{2.39}$$

where the d^{ij} are numbers. The partition function is then $|d|^2$; an eminently respectable topological invariant. Actually one can do better and establish that $d^{ij} = \delta^{ij}$, so that $Z_{G/G}$ is just the dimension of the space of holomorphic sections of $\mathcal{W}$, or the number of conformal blocks of the G WZW model [16]. This dimension can be calculated by explicit evaluation of the partition function [24].

The wavefunctions which include observables (2.29) are anti-holomorphic as their insertions introduce no extra metric dependence. Formally, also other gauge invariant observables will lead to conformal field theories. For example, the expectation value of a Wilson loop $\mathrm{Tr}_{R_i} P \exp \oint A$ will not spoil (2.34). On the other hand, the introduction of $\mathrm{Tr}_{R_i} P \exp \oint B$ will yield a wavefunction that does not satisfy (2.34). For the G/G model this implies that Wilson loops of A, in spite of the fact that they look like eminently respectable topological observables, do not lead to topological correlation functions. This can also be seen rather directly from the proof of metric independence of the partition function in [3] (or, for the topological Kazama-Suzuki models, in section 5.1 below), in which the A-equations of motion enter in a crucial way (e.g. in section 5.1 to establish the BRST-exactness of the energy momentum tensor).

Nevertheless, there are topological observables in the G/G model depending on A (and g), namely the images of 'horizontal' Wilson loops under the equivalence [24] of Chern-Simons theory on $\Sigma \times S^1$ with the G/G model on Σ [33].

2.5 $N = 1$ Coset Models

The general (anomalously) gauged WZW model $S(A, B; g)$ studied above has an $N = 1$ supersymmetric extension. To describe the action and the field content, let us orthogonally decompose

$$(\mathfrak{g}_{L,R})^{\mathbb{C}} = (\mathfrak{h}_{L,R})^{\mathbb{C}} \oplus (\mathfrak{k}_{L,R})^{\mathbb{C}} \ . \tag{2.40}$$

The supersymmetric extension has Weyl fermions ψ_- with values in $(\mathfrak{k}_L)^{\mathbb{C}}$ and ψ_+ with values in $(\mathfrak{k}_R)^{\mathbb{C}}$. The action is

$$S(A, B; \psi_+, \psi_-; g) = S(A, B; g) + \frac{i}{4\pi} \int_\Sigma \psi_- D_z(A) \psi_- + \frac{i}{4\pi} \int_\Sigma \psi_+ D_{\bar{z}}(B) \psi_+ . \tag{2.41}$$

The covariant derivatives are defined by

$$\begin{aligned} D_z(A)\psi_- &= \partial_z \psi_- + [A_z, \psi_-] \\ D_{\bar{z}}(B)\psi_+ &= \partial_{\bar{z}} \psi_+ + [B_{\bar{z}}, \psi_+]. \end{aligned} \tag{2.42}$$

The action enjoys the supersymmetry,

$$\begin{aligned} \delta g &= i\epsilon_+ \psi_- g + i\epsilon_- g \psi_+ \\ \delta \psi_- &= \epsilon_+ \Pi_L (D_{\bar{z}}(B) g . g^{-1} + i\psi_- \psi_-) \\ \delta \psi_+ &= \epsilon_- \Pi_R (g^{-1} D_z(A) g - i\psi_+ \psi_+), \end{aligned} \tag{2.43}$$

and $\Pi_{L,R}$ projects onto the $\mathfrak{k}_{L,R}$ part of the Lie algebra. These transformations are completely compatible with the gauge symmetry of the theory. In particular, this means that one can consider the usual gauging to arrive at the standard $N = 1$ action with $(A, B) = (A, A)$. One may also consider the axially gauged supersymmetric model $(A' + a, B' + b) = (A' + a, A' - a)$. However, care must be exercised as the chiral coupling to the fermions may produce an anomaly. One way around this is to use, in the $U(1)$ sector, a chirally preserving regularization from the outset.

When one takes $A = B$, (2.41) gives one a Lagrangian realization of the $N = 1$ super-GKO construction [5, 15]. It is useful to adopt a slightly different notation than that used in the above equations. In this situation A is a $\mathfrak{h} \equiv \mathrm{Lie}H$ valued gauge field for the (anomaly free) adjoint subgroup H of $G_L \times G_R$. $\psi_\pm$ are then Weyl fermions taking values in the complexification $\mathfrak{k}^{\mathbb{C}}$ of $\mathfrak{k}$, the orthogonal complement to $\mathfrak{h}$ in $\mathfrak{g} \equiv \mathrm{Lie}G$,

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{k} \ , \qquad \psi_\pm \in \mathfrak{k}^{\mathbb{C}} \ . \tag{2.44}$$

We will denote by $\Pi_{\mathfrak{h}}$ and $\Pi_{\mathfrak{k}}$ the orthogonal projectors onto $\mathfrak{h}$ and $\mathfrak{k}$ respectively. The $N = 1$ (actually $(1,1)$) supersymmetry in this formulation is,

$$\begin{aligned}
\delta g &= i\epsilon_- g\psi_+ + i\epsilon_+\psi_- g \\
\delta\psi_+ &= \epsilon_-\Pi_{\mathfrak{k}}(g^{-1}D_z g - i\psi_+\psi_+) \\
\delta\psi_- &= \epsilon_+\Pi_{\mathfrak{k}}(D_{\bar{z}}gg^{-1} + i\psi_-\psi_-) \\
\delta A &= 0\ .
\end{aligned} \tag{2.45}$$

2.6 Supersymmetric Wave Functions and Factorization

In order to establish the factorization of the general $N = 1$ supersymmetric WZW model, we follow the same procedure as in the bosonic coset model. Let us begin with the wavefunction

$$\Psi^{N=1}(A, B) = \int DgD\psi_-\ \exp -k\left(S(A, B; g) + \frac{i}{4\pi}\int_\Sigma \psi_- D_z(A)\psi_-\right). \tag{2.46}$$

The path integral enjoys the supersymmetry

$$\begin{aligned}
\delta g &= i\epsilon_+\psi_- g \\
\delta\psi_- &= \Pi_L\epsilon_+(D_{\bar{z}}(B)g.g^{-1} + i\psi_-\psi_-)
\end{aligned} \tag{2.47}$$

Unless certain conditions are met the wavefunction vanishes due to the presence of fermionic zero modes. When there are fermionic zero we should absorb them by introducing operators of the form of, say

$$\prod_{i=1}^{n} \psi_-(x_i) \tag{2.48}$$

where n is the number of ψ_- zero modes.

Gauge invariance is another story. Not only is the bosonic action not gauge invariant, but under a gauge transformation the Weyl determinant (i.e. the determinant of the Weyl fermions) picks up an anomaly. Consequently, one has

$$\left(D^A_\mu \frac{D}{DA_\mu} + \frac{i(k + c_G)}{4\pi}\epsilon^{\mu\nu}F_{\mu\nu}(A)\right)\Psi^{N=1}(A, B) = 0 \tag{2.49}$$

and hence a corresponding modification of the geometrical interpretation of the wave function. This equation picks up commutators on the righthand

side (with delta function support) when operator insertions are included to soak up zero modes.

We define a conjugate wavefunction by ($h \in G$)

$$\overline{\Psi^{N=1}(A,B)} = \int DhD\psi_+ \ \exp -k\left(S(B,A;h) + \frac{i}{4\pi}\int_\Sigma \psi_+ D_{\bar z}(A)\psi_+\right), \tag{2.50}$$

which is invariant under

$$\begin{aligned} \delta h &= i\epsilon_- h\psi_+ \\ \delta\psi_+ &= \Pi_R\epsilon_-(h^{-1}D_{\bar z}(B)h - i\psi_+\psi_+) \end{aligned} \tag{2.51}$$

Holomorphic factorization is the the statement that, with $B \in \mathfrak{g}$,

$$Z^{N=1} = |\Psi^{N=1}|^2 \tag{2.52}$$

When $B \in \mathfrak{g}$ the fermionic coupling to B in (2.41) vanishes as there are no ψ_+'s. Now,

$$\begin{aligned} |\Psi^{N=1}|^2 &= \int DADB\ \Psi^{N=1}(A,B)\overline{\Psi^{N=1}(A,B)} \\ &= \int DAD\psi\left(\int DBDlDh\, \mathrm{e}^{-kS(A,B;l) - kS(B,A;h)}\right)\mathrm{e}^{-kS(A,\psi)} \\ &= \int DAD\psi Dg\ \mathrm{e}^{-kS(A,g) - kS(A,\psi)} \end{aligned} \tag{2.53}$$

where we have used the factorization of the bosonic coset model in passing from the second to the third line and defined

$$S(A,\psi) = \frac{i}{4\pi}\int_\Sigma \psi_- D_z(A)\psi_- + \frac{i}{4\pi}\int_\Sigma \psi_+ D_{\bar z}(A)\psi_+. \tag{2.54}$$

This is the result that we are after. In order to see that the supersymmetry variations (2.43) come out right, we note that the variable g arises as $g = lh$ and thus has the transformation rules as in (2.43). The Gaussian integration over B is saturated by the equation of motion

$$B = -\frac{1}{2}[(i*-1)A^l - (i*+1)A^{h^{-1}}] \tag{2.55}$$

which means that the part of the ψ_- variation (2.47) involving the covariant derivative is

$$\begin{aligned} \delta\psi_- &= \Pi_L\epsilon_+(D_{\bar z}(B)l.l^{-1} \\ &= \Pi_L\epsilon_+(D_{\bar z}(A^{h^{-1}})l.l^{-1} \\ &= \Pi_L\epsilon_+(D_{\bar z}(A)g.g^{-1} \end{aligned} \tag{2.56}$$

which agrees with that in (2.43). Likewise, from (2.51)

$$\begin{aligned}\delta\psi_+ &= \Pi_R \epsilon_- h^{-1} D_z(B) h \\ &= \Pi_R \epsilon_- h^{-1} D_z(A^l) h \\ &= \Pi_R \epsilon_- g^{-1} D_z(A) g,\end{aligned} \tag{2.57}$$

again in agreement with (2.43). One can now understand the, rather perplexing, supersymmetry of the coset models as a symmetry on the left for l and on the right for h.

2.7 More Supersymmetric Models

Notice that one gets a perfectly respectable field theory in (2.53) even if B takes values in a subalgebra of $\mathfrak{g}$ and is not coupled to fermions. The resulting theory will be both supersymmetric and gauge invariant. The bosonic part of the theory is, as we noted before, a $(G \times G')/(H_L \times H_R)$ model. Supersymmetry requires that $T_e(G'/H) = T_e(G/H)$ for the left and right movers to match. This essentially sets $G = G'$ (up to discrete group actions). The supersymmetric model is therefore a $(G \times G)/[(H, H') \times (H', H)]$ theory with the left and right moving fermions taking values in $\mathfrak{g}/\mathfrak{h}$.

One may consider $B \in \mathfrak{h}'$ also to couple to fermions in a wavefunction with action (2.41). The gauge invariant and supersymmetric theory that one obtains on taking the norm is identical to the one described in the previous paragraph except that there are, in addition, left and right moving fermions with values in $\mathfrak{g}/\mathfrak{h}'$.

3 $N = 2$ Cosets and Topological Kazama-Suzuki Models

Now we have come to the main part of this paper, in which we will deal with $N = 2$ coset models and their topological partners. For the most part, we will be interested in the standard G/H supersymmetric models where G is a compact semi-simple Lie group (which we will also throughout assume to be simply laced), and H is a closed subgroup of G.

3.1 Lagrangian Realization of Kazama-Suzuki and More General $N=2$ Models

In [1], Kazama and Suzuki investigated (at the current algebra level), under which conditions on G and H the $N=1$ superconformal algebra of the coset model could be extended to an $N=2$ superconformal algebra. For our purposes, the most convenient characterization of the results is the following (see e.g. [1, 6, 34]). The coset model has an $N=2$ superconformal algebra iff there exists a direct sum decomposition

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{h}^{\mathbb{C}} \oplus (\mathfrak{k}^{+} \oplus \mathfrak{k}^{-}) \ , \tag{3.1}$$

such that

$$\begin{array}{ll} 1) & \dim \mathfrak{k}^{+} = \dim \mathfrak{k}^{-} \\ 2) & [\mathfrak{k}^{\pm}, \mathfrak{k}^{\pm}] \subset \mathfrak{k}^{\pm} \\ 3) & \mathrm{Tr}\,|_{\mathfrak{k}^{+}} = \mathrm{Tr}\,|_{\mathfrak{k}^{-}} = 0 \ . \end{array} \tag{3.2}$$

In fact, it can be seen rather directly that these conditions imply [16, 17] that the supersymmetry (2.45) of the action is enlarged to a $(2,2)$ supersymmetry. Namely, denoting the $\mathfrak{k}^{+}$-components of the fermions (ψ_{+}, ψ_{-}) as (α_{+}, β_{+}) and the $\mathfrak{k}^{-}$-components by (β_{-}, α_{-}) the action can be written as

$$kS_{KS}(g, A, \alpha, \beta) = kS_{G/H}(g, A) + \frac{ik}{2\pi}\int_{\Sigma} \beta_{-} D_{\bar{z}} \alpha_{+} + \beta_{+} D_{z} \alpha_{-} \ . \tag{3.3}$$

There is thus an R-symmetry of the fermionic part of the action with respect to which the fermions $\alpha_{\pm}$ and $\beta_{\pm}$ have charges ± 1 respectively, and the supersymmetry transformations may be split into their $\mathfrak{k}^{\pm}$-parts which are separate invariances of the action. For instance, for the left-movers one has [19]

$$\begin{aligned} \delta g &= i\bar{\epsilon}^{-} g \alpha_{+} + i\bar{\epsilon}^{+} g \beta_{-} \\ \delta \alpha_{+} &= \bar{\epsilon}^{+} \Pi_{+} (g^{-1} D_{z} g - i\alpha_{+}\beta_{-} - i\beta_{-}\alpha_{+}) - i\bar{\epsilon}^{-} \alpha_{+}\beta_{-} \\ \delta \beta_{-} &= \bar{\epsilon}^{-} \Pi_{-} (g^{-1} D_{z} g - i\alpha_{+}\beta_{-} - i\beta_{-}\alpha_{+}) - i\bar{\epsilon}^{-} \beta_{-}\alpha_{+} \\ \delta A &= \delta \alpha_{-} = \delta \beta_{+} = 0 \ . \ , \end{aligned} \tag{3.4}$$

where $\Pi_{\pm}$ denotes the projectors onto $\mathfrak{k}^{\pm}$ respectively. There is an analogous equation for the right-mnovers.

The conditions to be satisfied so that the $N=1$ supersymmetry is enhanced to an $N=2$ supersymmetry can also be met when the gauging is not

diagonal. In fact, when (3.2) is satisfied the action (2.41) has an $N = 2$ supersymmetry. Once more this means that there are $N = 2$ axially gauged models as well as a large class of non-diagonal coset $N = 2$ theories that come on 'squaring' (2.41). As these models possess the full $N = 2$ supersymmetry, they differ from the 'heterotic coset models' studied in [35] as generalizations of Kazama-Suzuki models. In the following, we will concentrate on the standard Kazama-Suzuki models.

The conditions (3.2) imply in particular that G/H is even dimensional so that $\mathrm{rk}\, G - \mathrm{rk}\, H = 2n$ is even. Regarding the decomposition $\mathfrak{k}^{\mathbb{C}} = \mathfrak{k}^{+} \oplus \mathfrak{k}^{-}$ as a decomposition of the complexified tangent space of G/H at the origin, (3.2) implies that G/H has an H-invariant complex structure, the (integrable) $\pm i$ eigenspaces of the complex structure corresponding to $\mathfrak{k}^{\pm}$ respectively. This means that we can regard $\mathfrak{k}^{-}$ as the complex conjugate of $\mathfrak{k}^{+}$.

While there are plenty of Kazama-Suzuki models with $\mathrm{rk}\, H < \mathrm{rk}\, G$ (e.g. based on even-dimensional groups G with H trivial [36, 37]), it appears that for most conformal field theory purposes one can restrict oneself to the case $\mathrm{rk}\, G = \mathrm{rk}\, H$ by the sequential G/H method of [1] (which permits one to write

$$G/H \sim G/(H \times U(1)^{2n}) \times U(1)^{2n} \tag{3.5}$$

at the level of symmetry algebras, expressing the given G/H $N = 2$ coset model as the product of a model with H of maximal rank and the well understood $N = 2$ theory based on $U(1)^{2n}$) combined with Abelian S-duality for the bosonic part of the action (see [37]).

If $\mathrm{rk}\, G = \mathrm{rk}\, H$, then H contains the maximal torus T of G and the conditions (3.2) are equivalent to the requirement that G/H be a Kähler manifold. By a theorem of Borel, H is then the centralizer of some (not necessarily maximal) torus of G. In that case, $\mathfrak{h}^{\mathbb{C}} \oplus \mathfrak{k}^{+}$ is a parabolic subalgebra of $\mathfrak{g}^{\mathbb{C}}$, and in terms of the Cartan decomposition

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{t}^{\mathbb{C}} \oplus_{\alpha} \mathfrak{g}_{\alpha} \oplus_{\alpha} \mathfrak{g}_{-\alpha} \ , \qquad \pm\alpha \in \Delta^{\pm}(G) \ , \tag{3.6}$$

the subalgebras $\mathfrak{k}^{\pm}$ can be realized as the sum of the root spaces associated with the roots

$$\Delta^{\pm}(G/H) = \Delta^{\pm}(G) \setminus \Delta^{\pm}(H) \tag{3.7}$$

of G which are not roots of H,

$$\mathfrak{k}^{\pm} = \oplus_{\alpha} \mathfrak{g}_{\pm\alpha} \ , \qquad \pm\alpha \in \Delta^{\pm}(G/H) \ . \tag{3.8}$$

If $\mathfrak{k}$ is a symmetric subalgebra of $\mathfrak{g}$, i.e. such that

$$[\mathfrak{k},\mathfrak{k}] \subset \mathfrak{h} \ , \tag{3.9}$$

which implies that the algbras $\mathfrak{k}^{\pm}$ are Abelian,

$$[\mathfrak{k}^{\pm},\mathfrak{k}^{\pm}] = 0 \ , \tag{3.10}$$

then G/H is automatically Kähler and is what is known as a hermitian symmetric space. These spaces are completely classified and examples are the complex Grassmannians $SU(m+n)/SU(m)\times SU(n)\times U(1)$ and the 'twistor spaces' $SO(2n)/SU(n)\times U(1)$.

It is Kazama-Suzuki models based on hermitian symmetric spaces which are in a sense the easiest to understand and which have received the most attention in the literature (see e.g. [1, 4, 7, 8, 9, 10]). They are also phenomenologically the most appealing models as they have no extra $U(1)$-symmetries beyond that dictated by the $N=2$ superconformal algebra [1] and thus lead to the minimal $E_6\times E_8$ gauge group via the Gepner construction.

Let us briefly return to the general case to introduce some more notation that we will require later on. We denote by c_G and c_H the dual Coxeter numbers of G and H, c_H being understood as a vector (c_i) when H has several simple factors H_i, and by ρ_G and ρ_H the Weyl vectors of G and H,

$$\rho_{G,H} = \tfrac{1}{2}\sum_{\alpha\in\Delta^+(G,H)} \alpha \ . \tag{3.11}$$

Their difference $\rho_{G/H}=\rho_G-\rho_H$ has the property that it is orthogonal to all the simple factors of H,

$$\mathrm{Tr}\,\rho_{G/H}\alpha = 0 \qquad \forall\,\alpha\in\Delta^+(H) \tag{3.12}$$

and lies in the direction generated by the $U(1)$-current of the $N=2$ superconformal algebra. In the hermitian symmetric models it is also the generator of the single $U(1)$-factor of H.

Upon bosonization the $N=2$ Kazama-Suzuki level k coset model can be described as the (bosonic) coset

$$[(G\times SO(\dim G/H)_1)/H]_k \ , \tag{3.13}$$

with $SO(\dim G/H)$ at level 1 representing the bosonized fermions, and the embedding of H into $SO(\dim G/H)$ being given by the isotropy representation of H on the tangent space of G/H. Here the level of H is given by

$k + c_G - c_H$ (for simply laced G). For more information on the Lie algebraic aspects of these models we refer to [11].

As the $N = 2$ theories are in particular $N = 1$ models one can adopt the wavefunctions of the previous section to establish holomorphic factorization in these theories as well. One can consider more 'refined' wavefunctions as well, which have explicit dependence on the fermion fields. However, these are not needed for our present purposes.

3.2 The Topological Twist of the Kazama-Suzuki Model

It is well known that an $N = 2$ superconformal field theory can be twisted to a topological conformal field theory, i.e. a theory with a BRST-exact and traceless energy momentum tensor with traceless superpartner [38, 39, 40]. In fact, consider the standard $N = 2$ superconformal algebra

$$\begin{aligned}
G^{\pm}(z)G^{\mp}(w) &= \frac{\frac{2}{3}c}{(z-w)^3} \pm \frac{2J(w)}{(z-w)^2} + \frac{2T(w) \pm \partial_w J(w)}{(z-w)} + \dots \\
J(z)G^{\pm}(w) &= \pm\frac{G^{\pm}(w)}{(z-w)} + \dots \\
J(z)J(w) &= \frac{\frac{1}{3}c}{(z-w)^2} + \dots \\
T(z)J(w) &= \frac{J(w)}{(z-w)^2} + \frac{\partial_w J(w)}{(z-w)} + \dots \\
T(z)G^{\pm}(w) &= \frac{\frac{3}{2}G^{\pm}(w)}{(z-w)^2} + \frac{\partial_w G^{\pm}(w)}{(z-w)} + \dots \\
T(z)T(w) &= \frac{\frac{1}{2}c}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial_w T(w)}{(z-w)} + \dots \ .
\end{aligned} \tag{3.14}$$

It follows that the twisted energy momentum tensor

$$T_{top}(z) = T(z) \pm \tfrac{1}{2}\partial_z J(z) \tag{3.15}$$

satisfies a Virasoro algebra with central charge $c_{top} = 0$, that (for the +-sign in (3.15)) the conformal weights of G^+ and G^- have shifted from their original value $\frac{3}{2}$ to 1 and 2 respectively, and that

$$T_{top}(z) = \{Q, G^-(z)\} \ , \tag{3.16}$$

where

$$Q = \oint G^+(z)dz \quad , \qquad Q^2 = 0 \quad . \tag{3.17}$$

The physical states of this twisted theory, defined as the cohomology of Q, can be represented by the chiral primary fields [4] of the original $N = 2$ superconformal field theory.

In fact, as indicated there are two possible twistings, the second corresponding to changing the sign of J, and upon putting together left and right movers, one can obtain two inequivalent topological field theories, known as the A and B models respectively [41]. In the case of $N = 2$ superconformal (i.e. Calabi-Yau) sigma models, the $N = 2$ $U(1)$-current J is the R-current of the theory and the twists (3.15) can be mimicked at the level of the action by adding a term of the form

$$\frac{1}{2}(\omega_z J_{\bar{z}} \pm \omega_{\bar{z}} J_z) \tag{3.18}$$

to the Lagrangian, where ω is the spin-connection. The whole effect of this term is to change the spins of the fermions form $1/2$ to 0 or 1. In particular, the A-model then coincides with the topological sigma model introduced in [38].

In the case of Kazama-Suzuki models, the $N = 2$ $U(1)$-current is more complicated. At the current algebra level, it is given by the sum of the $\rho_{G/H}$-component of the H-current (itself having a bosonic and a fermionic contribution) and the fermion number operator [1]. Hence, in this case it is certainly not correct to try to implement (3.15) by adding the term (3.18) to the Lagrangian (3.3). Nevertheless, by analogy with the topological sigma model, there is an obvious guess as to what the action of the A-model should be, namely the action (3.3) where one regards the fields α_+ and α_- as Grassmann odd scalars, and β_- and β_+ as anti-commuting $(1,0)$ and $(0,1)$ forms respectively. This is tantamount to shifting the gauge field by half the spin connection.

It has been shown by Nakatsu and Sugawara [18] that this 'twist', which amounts to adding to the action (3.3) a term of the form (3.18) with J replaced by the R-current of the Kazama-Suzuki model, indeed reproduces the A-twist (3.15) of the energy momentum tensor by the $N = 2$ $U(1)$-current J at the conformal field theory level. We know of no short-cut that would establish this directly and refer to the analysis [18] for the details of the argument.

We will, however, show in section 5 that the energy momentum tensor of the theory defined by this twisted Lagrangian is BRST exact (modulo the equations of motion of the gauge field). That it differs from the energy-momentum tensor of the action (3.3) by the derivative of the R-current is obvious by construction since all one has changed is the spin of the fermionic fields.

The B-model, on the other hand, can be obtained by an analogous twisting of a suitable axially gauged $N = 2$ model. In general, the B-model is of interest, in particular in relation with issues like mirror symmetry [41, 20]. However, as the (a, c) ring of many of the Kazama-Suzuki models we will consider in the following and in [21] is known to be trivial [4] (unless one considers orbifold models), we will not pursue this here.

3.3 Basic Properties of the Topological Kazama-Suzuki Model

It follows from the above that the topological Kazama-Suzuki model can be described by the action

$$S_{TKS}(g, A, \alpha^+, \bar{\alpha}^-, \beta_z^-, \bar{\beta}_{\bar{z}}^+) = S_{G/H}(g, A) + \frac{1}{2\pi}\int_\Sigma \beta_z^- D_{\bar{z}}\alpha^+ + \bar{\beta}_{\bar{z}}^+ D_z \bar{\alpha}^- \quad . \tag{3.19}$$

Here α^+ and $\bar{\alpha}^-$ are Grassmann odd scalars taking values in $\mathfrak{k}^+$ and $\mathfrak{k}^-$ respectively, β_z^- is a Grassmann odd $\mathfrak{k}^-$-valued $(1,0)$-form and $\bar{\beta}_{\bar{z}}^+$ a Grassmann odd $\mathfrak{k}^+$-valued $(0,1)$-form and we have absorbed a factor of i into the definition of α^+ and $\bar{\alpha}^-$ in order to prevent a proliferation of i's in subsequent equations. Also, for notational ease we will from now on mostly not indicate explicitly the form and Lie algebra labels and denote the fields simply by $(\alpha, \beta, \overline{\alpha}, \overline{\beta})$.

The action is invariant under two BRST-like transformations Q and $\overline{Q}$ with $\delta = Q + \overline{Q}$, the remnants of the original $N = 2$ supersymmetry and the counterparts of the left- and right-moving BRST tranformations of the A-twist of the corresponding conformal field theory. These transformations are

$$\begin{aligned} Qg &= g\alpha \\ Q\alpha &= -\alpha^2 \\ Q\beta &= \Pi_-(g^{-1}D_z g - [\alpha, \beta]) \\ Q(\text{rest}) &= 0 \end{aligned}$$

$$
\begin{aligned}
\overline{Q}g &= \overline{\alpha}g \\
\overline{Q}\overline{\alpha} &= \overline{\alpha}^2 \\
\overline{Q}\overline{\beta} &= \Pi_+(D_{\bar{z}}gg^{-1} + [\overline{\alpha},\overline{\beta}]) \\
\overline{Q}(\text{rest}) &= 0
\end{aligned}
\tag{3.20}
$$

It follows from (3.20) that both Q and $\overline{Q}$ are nilpotent,

$$
Q^2 = \overline{Q}^2 = 0 \ . \tag{3.21}
$$

On the other hand, δ is only nilpotent on-shell. From the above one finds that $\delta^2 = [Q, \overline{Q}]$ acts as

$$
\begin{aligned}
&[Q,\overline{Q}]g = [Q,\overline{Q}]\alpha = [Q,\overline{Q}]\overline{\alpha} = 0 \ , \\
&[Q,\overline{Q}]\beta = \Pi_-(g^{-1}D_z\overline{\alpha}g) \ , \qquad [Q,\overline{Q}]\overline{\beta} = \Pi_+(gD_{\bar{z}}\alpha g^{-1}) \ ,
\end{aligned}
\tag{3.22}
$$

so that $\delta^2 = 0$ modulo the β equations of motion.

One other interesting property of these transformations is, that they can be decomposed further into the sum of a standard nilpotent BRST like symmetry s acting only on the fermionic part of the action and a 'topological' symmetry Q_T, acting only on g and β. Both of these are separate invariances of the action. E.g. one has $Q = s + Q_T$ with

$$
\begin{aligned}
&s\alpha = -\alpha^2 \ , \qquad s\beta = -\Pi_-[\alpha,\beta] \\
&Q_T g = g\alpha \ , \qquad Q_T\beta = \Pi_-(g^{-1}D_z g) \\
&s(S_{TKS}) = Q_T(S_{TKS}) = s^2 = 0 \ .
\end{aligned}
\tag{3.23}
$$

The topological symmetry Q_T only squares to zero equivariantly, as is familiar from cohomological field theories and s acts trivially if G and H are such that G/H is hermitian symmetric. In that case, the terms quadratic in the Grassmann odd fields are absent from the right-hand side of (3.20).

For later use we record here the A-variation of the action of the A-model,

$$
\delta_A S_{TKS} = \tfrac{1}{2\pi}\int_\Sigma (J_z - [\alpha,\beta])\delta A_{\bar{z}} + (J_{\bar{z}} - [\overline{\alpha},\overline{\beta}])\delta A_z \ , \tag{3.24}
$$

where

$$
J_z = g^{-1}D_z g \ , \qquad J_{\bar{z}} = -D_{\bar{z}}gg^{-1} \ , \tag{3.25}
$$

are the G-currents and of course only their $\mathfrak{h}$-parts $\Pi_{\mathfrak{h}}(J_z) \equiv J_z^{\mathfrak{h}}$ and $J_{\bar{z}}^{\mathfrak{h}}$ contribute in (3.24).

3.4 Wave Functions and Holomorphic Factorization

The twisted wavefunction is defined by

$$\Psi^{TKS}(A,B) = \int Dg\, \mathrm{e}^{-kS_T(A,B;\overline{\beta},\overline{\alpha};g)} \ , \tag{3.26}$$

where

$$S_T(A,B;\overline{\beta},\overline{\alpha};g) = S(A,B;g) + \frac{i}{2\pi}\int_\Sigma \overline{\beta} d_A \overline{\alpha}. \tag{3.27}$$

The action S_T is invariant under the two sets of transformations

$$\begin{aligned} \overline{Q}_T g &= \overline{\alpha} g \\ \overline{Q}_T \overline{\beta} &= \Pi_+\left(\frac{(1+i*)}{2} d_B g.g^{-1}\right) \end{aligned} \tag{3.28}$$

and

$$\begin{aligned} \overline{s}\,\overline{\alpha} &= \overline{\alpha}^2 \\ \overline{s}\overline{\beta} &= \Pi_+[\overline{\alpha},\overline{\beta}] \end{aligned} \tag{3.29}$$

(all other transformations are zero).

Arguments, familiar from previous sections, tell us that this wavefunction is anti-holomorphic in the appropriate sense

$$\nabla^{(1,0)}\Psi^{TKS}(A,B,g) = 0. \tag{3.30}$$

What is not apparent, at first sight, is that Ψ^{TKS} is also holomorphic,

$$\nabla^{(0,1)}\Psi^{TKS}(A,B;g) = 0. \tag{3.31}$$

To establish this one notes that

$$\nabla^{(0,1)}.\mathrm{e}^{-kS_T} = -[\frac{k}{4\pi}\overline{Q}\int_\Sigma \delta * \overline{\beta}\left(d_B g.g^{-1} + \frac{1}{2}[\overline{\alpha},\overline{\beta}]\right)]\mathrm{e}^{-kS_T}. \tag{3.32}$$

A similar calculation will be performed when we discuss the metric independence and the coupling to topological gravity in more detail. There we will be more explicit about the derivation.

By the BRST invariance of the theory the expectation value of $\overline{Q}$ exact terms is zero. Hence the right hand side of (3.32) vanishes in the path integral and we are thus led to the fact that the wavefunction is anti-holomorphic as

well. As the proof of factorization goes through *verbatim* as well (leading, as in the bosonic case, to the conclusion that the partition function is metric independent) this is about the best that one could hope for in a topological (conformal) field theory.

So far we have considered topological models coming from a diagonal coset. However, one may establish the topological nature of a more general class of theories. E.g. when both G/H and G/H' are Kähler one can form a twisted version of the, now $N = 2$ invariant, action (2.41), which we denote by $\hat{S}_{TKS}$. The wavefunctions, $\hat{\Psi}$, obtained on using $\hat{S}_{TKS}$ as the action are indeed both holomorphic and anti-holomorphic. One arrives at this conclusion by the same reasoning we used to show that Ψ^{TKS} is holomorphic.

4 Localization and Abelianization

In [16], Witten has shown that the BRST symmetry of the topological $\mathbb{CP}(1)$ Kazama-Suzuki model can be used to localize the theory to a bosonic $U(1)/U(1)$ model (i.e. a compact Abelian BF theory) with some quantum corrections and selection rules arising from the chiral anomaly of the fermionic sector. Here we will show that, in general, a G/H model can be localized to a perturbed H/H model. Heuristically speaking, the BRST symmetry permits one to linearize the G/H-part of the bosonic action, and up to a chiral anomaly the resulting determinant cancels against that arising form the integration over the non-zero-modes of the Grassmann odd fields (taking values in $(\mathfrak{g}/\mathfrak{h})^{\mathbb{C}}$).

By the results of [24], this theory in turn can be further localized (Abelianized) to a T_H/T_H model (T_H a maximal torus of H), permitting one to reduce the task of calculating correlation functions in the non-linear, non-Abelian and supersymmetric topological Kazama-Suzuki model to a calculation of correlators in an Abelian bosonic topological field theory. As we will show in [21], this drastic simplification of the problem permits one to be quite explicit about the structure of the ring of observables in these models and makes calculations rather straightforward in some cases.

4.1 Localization: General Considerations

The idea behind the localization of the path integral is that, taking Q to be a BRST operator, the Grassmann odd fields can be thought of as ghosts.

Thinking about them in this way leads to a 'cancellation' between them and would be physical modes. By the supersymmetry (3.20), one sees that the pairing is between certain components of g and α and $\overline{\alpha}$. Essentially one will be able to eliminate the 'G/H' parts of g. This is how it works in principle. Unfortunately, in practice things are somewhat more complicated. Rather than finding a straightforward cancellation one finds various possible 'branches' for the cancellation to take place. The reason for this plethora of possibilities rests in the manner in which one projects onto the paired modes.

There are various ways to establish localization in topological theories, e.g. formulated as a BRST fixed point theorem as in [16]. Here we present an alternative argument which has the virtue of making the localization quite explicit. Namely, one can add to the action (3.19) a Q-exact term enforcing the localization. Consider, for example,

$$\begin{aligned} Q(\beta g^{-1} D_{\bar{z}} g) &= \Pi_-(g^{-1} D_z g)\Pi_+(g^{-1} D_{\bar{z}} g) - \Pi_-([\alpha,\beta])\Pi_+(g^{-1} D_{\bar{z}} g) \\ &- \beta D_{\bar{z}}(A^g)\alpha \ , \end{aligned} \tag{4.1}$$

and add this to the action with an arbitrary coefficient t,

$$S_{TKS} \rightarrow S_{TKS} + \tfrac{t}{2\pi} Q \int_\Sigma \beta_z g^{-1} D_{\bar{z}} g \ . \tag{4.2}$$

The partition function and BRST invariant observables do not depend on t (an easy way to see this is that differentiating with respect to t leads one to evaluate the expectation value of a BRST exact correlation function, which vanishes by virtue of the BRST invariance of the theory). One is free to take various limits, $t \rightarrow 0$ giving the original action, while in the limit $t \rightarrow \infty$ the semiclassical approximation becomes exact. The additional term preserves the other symmetries of the theory.

We wish to take the $t \rightarrow \infty$ limit. In this limit the main contribution to the path integral will come around configurations for which the additional term (4.1) vanishes. The configurations of interest for us are

$$\begin{aligned} \Pi_- g^{-1} D_z g &= 0 \\ \Pi_+ g^{-1} D_{\bar{z}} g &= 0. \end{aligned} \tag{4.3}$$

There are various branches of solutions to these equations. One, which we shall call the main branch, corresponds to $g \in H$ for which (4.3) is automatically satisfied. This branch is algebraic, meaning that no differential

equations are taken to be satisfied. This is the only branch with this property. Write

$$g = \mathrm{e}^{i\phi} = \mathrm{e}^{i(\phi^h + \phi^+ + \phi^-)}. \tag{4.4}$$

The main branch corresponds to a cancellation between ϕ^+ and α^+ and between ϕ^- and $\overline{\alpha}^-$. In performing the path integral, the configuration $g \in H$ will correspond to a 'background' field, while $\phi^\pm$ are the quantum fields.

There are, however, many other branches. With the help of the Duhamel formula

$$\delta \mathrm{e}^X = \mathrm{e}^X \int_0^1 ds \mathrm{e}^{-sX} \delta X \mathrm{e}^{sX} \quad , \tag{4.5}$$

one has

$$\begin{aligned} \Pi_- \left(\frac{\mathrm{Ad}(\mathrm{e}^{i\phi}) - 1}{\mathrm{ad}(\phi)} \right) D_z \phi &= 0 \\ \Pi_+ \left(\frac{\mathrm{Ad}(\mathrm{e}^{i\phi}) - 1}{\mathrm{ad}(\phi)} \right) D_{\bar{z}} \phi &= 0. \end{aligned} \tag{4.6}$$

For example, for the hermitian symmetric spaces, there is a branch with $\phi^h = 0$ and $D_z \phi^- = D_{\bar{z}} \phi^+ = 0$. These equations can only be satisfied if certain topological criteria are met. This branch will or will not appear depending on the topology whereas the main branch is always present.

Even presuming that the bundle is such that these types of solutions are allowed, as long as we do not consider observables that include α, this branch cannot contribute, for if there are such ϕ then there will be α zero-modes.

4.2 Localization onto the Main Branch

When $t \to \infty$ the theory localizes onto solutions to $\Pi_- g^{-1} D_z g = 0$. In principle, for all the branches to be taken into account we would have to perform a background field expansion around each branch and in some way handle the fact that the different branches intersect. See [16] for the corresponding discussion in the case of $\mathbb{CP}(1)$ models, where there is only a rather small number of possible branches, and where by explicit analysis it can be argued that only the main branch contributes.

Here, we have no way of (or, at least, we have not succeeded in) eliminating all the other possible branches in full generality. Rather, we will proceed with

the tacit assumption that only the main branch, with $g \in H$, contributes to the evaluation of the path integral and expectation values of the operators that are of interest to us.

When dealing with the localized theory one gets more than just the classical configurations, of course, as there are one loop contributions to be taken into account as well. To implement this we consider the following scaling of the bosonic fields $\phi^{\pm}$ and the non-zero-modes of the fermionic fields,

$$\begin{aligned}(\phi^+, \phi^-) &\rightarrow (t^{-1/2}\phi^+, t^{-1/2}\phi^-) \\ (\alpha, \overline{\alpha}) &\rightarrow (t^{-1/2}\alpha, t^{-1/2}\overline{\alpha}) \\ (\beta, \overline{\beta}) &\rightarrow (t^{-1/2}\beta, t^{1/2}\overline{\beta}) \ .\end{aligned} \tag{4.7}$$

This scaling has unit Jacobian as the transformation of the bosonic fields is precisely compensated by that of the Grassmann odd scalars.

On using the Duhamel formula one then finds that

$$\begin{aligned}\Pi_- g^{-1}D_z g &\rightarrow iD_z(A^h)\tilde{\phi}^- + O(t^{-1/2}) \ , \\ \Pi_+ g^{-1}D_{\bar{z}} g &\rightarrow iD_{\bar{z}}(A^h)\tilde{\phi}^+ + O(t^{-1/2}) \ ,\end{aligned} \tag{4.8}$$

where $h = \exp i\phi^h$ and the twisted fields $\tilde{\phi}^{\pm} \in \mathfrak{k}^{\pm}$ are defined by

$$\tilde{\phi}^{\pm} = \int_0^1 ds \mathrm{e}^{-is\phi^h}\phi^{\pm}\mathrm{e}^{is\phi^h} = i\left(\frac{\mathrm{e}^{-i\,\mathrm{ad}\,\phi^h} - 1}{\mathrm{ad}\,\phi^h}\right)\phi^{\pm}. \tag{4.9}$$

One question that arises is, what happens to the path integral measure in passing from the group G valued field g to the group H valued field h and the G/H coset fields $\tilde{\phi}^{\pm}$? On making use of the Duhamel formula, we find that

$$Dg = \mathfrak{j}_{\mathfrak{g}}(\phi)D\phi \tag{4.10}$$

where

$$\mathfrak{j}_{\mathfrak{g}}(\phi) = \mathrm{Det}_{\mathfrak{g}}\left(\frac{1 - \mathrm{e}^{-\,\mathrm{ad}\,i\phi}}{i\,\mathrm{ad}\,\phi}\right). \tag{4.11}$$

Now on passing to the $\tilde{\phi}^{\pm}$ fields,

$$Dg = \mathfrak{j}_{\mathfrak{g}}(\phi)[j_{\mathfrak{g}/\mathfrak{h}}(\phi^h)]^{-1}D\phi^h D\tilde{\phi}^+ D\tilde{\phi}^- \tag{4.12}$$

with

$$\mathfrak{j}_{\mathfrak{g}/\mathfrak{h}}(\phi^h) = \mathrm{Det}_{\mathfrak{g}/\mathfrak{h}}\left(\frac{\mathrm{e}^{-i\,\mathrm{ad}\,\phi^h} - 1}{\mathrm{ad}\,\phi^h}\right). \tag{4.13}$$

Thus, the following things happen as $t \to \infty$:

- The bosonic action $S_{G/H}(g,A)$ reduces to the action $S_{H/H}(h,A)$ of the bosonic topological H/H model.
- The second term of (4.1) vanishes.
- The term $\beta D_{\bar{z}}\alpha$ of the original action disappears. It is replaced by the third term of (4.1), which reduces to $\beta D_{\bar{z}}(A^h)\alpha$. The other fermionic term remains unchanged.
- The first term in (4.1) only receives a contribution form the order $t^{-1/2}$ terms (4.8) of $\Pi_-(g^{-1}D_z g)$ and $\Pi_+(g^{-1}D_{\bar{z}}g)$. Thus the kinetic term for $\tilde{\phi}^{\pm}$ is
$$D_z(A^h)\tilde{\phi}^- D_{\bar{z}}(A^h)\tilde{\phi}^+ \quad .$$
- The path integral measure becomes $Dg = \mathrm{j}_{\mathfrak{h}}(\phi^h)D\phi^h D\tilde{\phi}^+ D\tilde{\phi}^- = DhD\tilde{\phi}^+ D\tilde{\phi}^-$ (the t dependence of the scaling has already been cancelled against the scaling of the Grassmann odd scalars).

Putting everything together one finds that the $t \to \infty$ limit of (4.2) is

$$S_{H/H}(h,A) + \tfrac{1}{2\pi}\int_\Sigma \beta D_{\bar{z}}(A^h)\alpha + \overline{\beta}D_z(A)\overline{\alpha} + D_z(A^h)\tilde{\phi}_- D_{\bar{z}}(A^h)\tilde{\phi}_+ \quad , \tag{4.14}$$

with the canonical measure for all the fields. Were it not for the fact that it is the operator $D_z(A)$ rather than $D_z(A^h)$ which acts on $\overline{\alpha}$, the determinants from the bosonic and fermionic terms would cancel modulo questions related to β zero modes. This can be made a little bit more explicit by adding a term $\epsilon\beta\overline{\beta}$ to the action and integrating out β and $\overline{\beta}$. If there are no β zero modes, then integrating over the α first implies $\beta = \overline{\beta} = 0$, so that everything is independent of ϵ. If there are β-zero modes, then these have to be soaked up and this can be accomplished by adding an analogous term $(\beta)^0(\overline{\beta})^0$ to the action (see section 6.4) and the argument still goes through. Anyway, upon integrating out the β's, one finds that the G/H-part of the action reduces to

$$\tilde{\phi}^+ D_{\bar{z}}(A^h)D_z(A^h)\tilde{\phi}^- + \alpha D_{\bar{z}}(A^h)D_z(A)\overline{\alpha} \quad . \tag{4.15}$$

Thus, finally, using the gauge invariance of the chiral anomaly and the H/H action ($S_{H/H}(h,A) = S_{H/H}(h,A^h)$) to send A^h to A, one finds that the

G/H-part of the topological Kazama-Suzuki model (the transverse part of the partition function) reduces to the ratio of determinants

$$Z_{\rm trans} = {\rm e}^{-W(A,h)} = \frac{{\rm Det}_{\mathfrak{k}}[D_{\bar{z}}(A)D_z(A^{h^{-1}})]}{{\rm Det}_{\mathfrak{k}}[D_{\bar{z}}(A)D_z(A)]} \quad . \tag{4.16}$$

The general structure of the effective action $W(A,h)$, as determined by the chiral anomalies, is as follows. Let us write the gauge group H as a product of (semi-) simple factors H_i and an Abelian part $U(1)^l$. Then each of the factors H_i gives rise to an H_i/H_i model at level $c_G - c_i$ where c_i is the dual Coxeter number of H_i. Combining this with the level k H_i/H_i actions which are the remnants of the bosonic G/H action after localization, one thus obtains, for each factor H_i, an H_i/H_i action at level $k + c_G - c_i$. This is in agreement with the current algebra (coset model) description of the Kazama-Suzuki models as (cf. section 3) the coset

$$(G_k \times SO_1(\dim G/H))/H_{k+c_G-c_H} \quad . \tag{4.17}$$

Likewise the gauge field coupling of the $U(1)$-factors is given by a level c_G ($c_{U(1)} = 0$) $U(1)/U(1)$ model (i.e. essentially a compact Abelian BF theory). However, because in the twisted model we are dealing not with fermions but with a $(1,0)$ system, there is an additional coupling to the scalar curvature R of the spin-connection implicit in (3.19). This coupling takes the form ($h = \exp i\phi^h$)

$$S_{TKS}^{\rm grav} = -\tfrac{i}{2\pi}\int_\Sigma {\rm Tr}\,\rho_{G/H}\phi^h R \ , \tag{4.18}$$

where R is normalized such that

$$\tfrac{1}{2\pi}\int_\Sigma R = \chi(\Sigma) \tag{4.19}$$

is the Euler number of Σ ($(2-2g)$ for a genus g surface). As $\rho_{G/H}$ is orthogonal to $\times_i H_i$, it is indeed only the $U(1)$-factors of H which contribute to this expression.

Altogether one finds that the effective action one obtains after localization and integration over the coset valued fermionic and bosonic fields is

$$S_{TKS}^{eff,k} = \sum_i (k + c_G - c_i) S_{H_i/H_i} + (k + c_G) S_{U(1)^l/U(1)^l} + S_{TKS}^{\rm grav} \quad . \tag{4.20}$$

It should be borne in mind that, in order to arrive at this action, we have integrated out only the non-zero-modes of the fermionic fields, so that the

integration over the zero modes is still to be performed. This is best understood within the context of specific examples, and we will describe this for the hermitian symmetric models in [21].

Localization can also be performed at the level of wave functions and it is readily checked that the result of localizing onto the main branch, at the level of the wave functions, reproduces the effective action $S_{TKS}^{eff,k}$.

4.3 Abelianization

If the gauge group H is Abelian (e.g. when one is dealing with the flag manifolds G/T, T a maximal torus of G), then the above localization establishes directly that calculations of correlation functions can be reduced to calculations in an Abelian topological field theory, thus significantly simplifying that task.

Something analogous can be achieved in the general non-Abelian case as well. Indeed, in [24] it was shown that the H_i gauge symmetry of the H_i/H_i model can be used to abelianize the theory, i.e. to reduce it to a T_i/T_i model (plus quantum corrections) where T_i is the maximal torus of H_i. Combining this with the $S_{U(1)^l/U(1)^l}$ action, one thus obtains a T_H/T_H model, where T_H is the maximal torus of H. In particular, therefore, if we are dealing with the Kähler models with $\mathrm{rk}\, G = \mathrm{rk}\, H$, then the Kazama-Suzuki model can be reduced to an Abelian T/T model (with the above mentioned quantum corrections and the gravitational coupling (4.18)).

These quantum corrections are of two kinds. The first one is a shift of the level of the H_i/H_i action by c_i so that

$$S_{TKS}^{eff,k} \to (k + c_G) S_{T_H/T_H} + S_{TKS}^{\mathrm{grav}} \quad . \tag{4.21}$$

The second correction is a finite-dimensional determinant arising from the ratio of the functional determinant from the H_i/T_i components of the gauge field and the Jacobian (Faddeev-Popov determinant) from the change of variables (choice of gauge) $h_i \to t_i \in T_i$. It can be written as

$$\exp\left[\tfrac{1}{4\pi}\int_\Sigma R \log\det(1 - \mathrm{Ad}(t_i))\right] \quad , \tag{4.22}$$

where the determinant is taken on the orthogonal complement of the Lie algebra of T_i in the Lie algebra of H_i. While we refer to [24] for a detailed derivation, let us make the following comments here.

1. The determinant is what is known as the Weyl determinant $\Delta_W^{H_i}(t_i)$ appearing in the finite-dimensional Weyl integral formula relating integrals of class functions on H_i to integrals over T_i.

2. As we are dealing with a topological theory, we can treat the fields t_i as position independent (in fact, explicit calculations show that only the constant modes of t_i contribute to the path integral).

3. Hence we can say that the localized and abelianized theory is defined by the action (4.21) with the measure for $t \in T_H$ modified to

$$D[t] \rightarrow D[t](\Delta_W^H(t))^{\chi(\Sigma_g)/2} \quad . \tag{4.23}$$

We have thus managed to reduce the original non-Abelian supersymmetric topological Kazama-Suzuki model to a much more tractable bosonic Abelian topological field theory, the entire information on the coset valued fields being encoded in the shifted levels and the gravitational coupling, and the Weyl determinant keeping track of the originally non-Abelian nature of the theory.

In performing actual calculations in the model described above, there are some further things that are good to keep in mind, for instance that use of the infinite-dimensional version of the Weyl integral formula employed above engenders [26] a sum over all isomorphism classes of T_i-bundles on Σ [24]. This as well as questions related to chiral anomalies, fermionic zero modes and the ensuing selection rules for correlation functions, which are more or less immediate analogues of those considered by Witten [16] for the $\mathbb{CP}(1)$ model, will be explained for the hermitian symmetric models in [21].

5 Coupling to Topological Gravity

In this section we will construct the coupling to topological gravity of the topological Kazama-Suzuki models introduced above. As a preliminary first step we prove by direct calculation that the partition function of the pure matter theory is indeed metric independent and that the energy-momentum tensor is BRST exact modulo the gauge field equations of motion (i.e. modulo the gauge currents). With this in hand one can, to some extent, rationalize the form of the theory coupled to topological gravity.

5.1 Metric Independence of the Partition Function

The topological nature of the theory defined by the action (3.19) follows (indirectly) from the fact that at the conformal field theory level this theory is equivalent to the A-twist of the Kazama-Suzuki model. It can also be inferred from the fact that the wave functions are both holomorphic and anti-holomorphic. However, it is instructive to establish this metric independence directly at the level of the Lagrangian realization we have chosen (this derivation also provides some of the steps required in showing that the wave functions are anti-holomorphic).

As usual, the energy-momentum tensor is defined as the variation of the action with respect to the metric $\rho_{\mu\nu}$, i.e.

$$\delta_\rho S = \int \sqrt{\rho}\delta\rho^{\mu\nu}T_{\mu\nu} \quad . \tag{5.1}$$

Applied to the action (3.19), one finds that (as expected) $T_{\mu\nu}$ is traceless, $T_{z\bar{z}} = 0$ and that the non-vanishing components are

$$\begin{aligned} 2\pi T_{zz} &= -\tfrac{1}{2}\operatorname{Tr} J_z J_z + \operatorname{Tr}\beta_z^- D_z\alpha^+ \\ &= -\tfrac{1}{2}\operatorname{Tr} J_z^{\mathfrak{h}} J_z^{\mathfrak{h}} - \operatorname{Tr} J_z^+ J_z^- + \operatorname{Tr}\beta_z^- D_z\alpha^+ \end{aligned} \tag{5.2}$$

$$2\pi T_{\bar{z}\bar{z}} = -\tfrac{1}{2}\operatorname{Tr} J_{\bar{z}}^{\mathfrak{h}} J_{\bar{z}}^{\mathfrak{h}} - \operatorname{Tr} J_{\bar{z}}^+ J_{\bar{z}}^- + \operatorname{Tr}\bar{\beta}_{\bar{z}}^+ D_{\bar{z}}\bar{\alpha}^- \tag{5.3}$$

Here J_z and $J_{\bar{z}}$ are the bosonic currents, $J_z = g^{-1}D_z g$, $J_{\bar{z}} = -D_{\bar{z}} g g^{-1}$, and $J^{\mathfrak{h}}$ and $J^{\pm}$ denote their components in $\mathfrak{h}$ and $\mathfrak{k}^{\pm}$ respectively.

The bosonic part of the energy momentum tensor is just of the (covariantized) Sugawara form, while the fermionic part is the standard energy-momentum tensor of a $(1,0)$-system.

We will now establish that, modulo the $A_{\bar{z}}$-equation of motion, the left-moving energy momentum tensor T_{zz} is Q-exact. Completely analogously, one can establish that $T_{\bar{z}\bar{z}}$ is $\overline{Q}$-exact, where we have used the decomposition $\delta = Q + \overline{Q}$ of the BRST symmetry into its left- and right-moving parts. By standard arguments, this then establishes the metric independence of the partition function and suitable correlation functions (e.g. correlation functions of BRST invariant and metric and A-independent operators).

Let us recall the action of $Q = Q_T + s$ on the fields,

$$\begin{aligned} Qg &= g\alpha \\ Q\alpha^+ &= -\tfrac{1}{2}[\alpha^+, \alpha^+] \end{aligned}$$

$$\begin{aligned} Q\beta_z^- &= J_z^- - \Pi_-[\alpha^+,\beta_z^-] \\ Q(\mathrm{rest}) &= Q^2 = 0 \ . \end{aligned} \tag{5.4}$$

(here s gives rise to the terms quadratic in the fermions). and the $A_{\bar{z}}$ equation of motion expressing the vanishing of the H gauge current,

$$J_z^{\mathfrak{h}} = \Pi_{\mathfrak{h}}[\alpha^+,\beta_z^-] \ . \tag{5.5}$$

One can use this equation to replace the term in (5.2) quadratic in the bosonic part of the gauge current by a quartic fermionic term,

$$J_z^{\mathfrak{h}} J_z^{\mathfrak{h}} \to \Pi_{\mathfrak{h}}[\alpha^+,\beta_z^-]\Pi_{\mathfrak{h}}[\alpha^+,\beta_z^-] \ . \tag{5.6}$$

This term, which vanishes identically for the hermitian symmetric models, is in general s-exact,

$$s(\mathrm{Tr}\,\beta_z^-[\alpha^+,\beta_z^-]) = \mathrm{Tr}\,\Pi_{\mathfrak{h}}[\alpha^+,\beta_z^-]\Pi_{\mathfrak{h}}[\alpha^+,\beta_z^-] \ . \tag{5.7}$$

The fact that this part of the metric variation can be cancelled by an expression quadratic in the gauge currents we have encountered before in the guise of the operator $\nabla^{(1,0)}$ (cf. (2.34)).

Furthermore, one can replace J_z^- by $Q_T\beta_z^-$. Proceeding in this way, one finds that

$$2\pi T_{zz} = -Q\,\mathrm{Tr}\,\beta_z^-(J_z^+ - \tfrac{1}{2}[\alpha^+,\beta_z^-]) \ . \tag{5.8}$$

To establish this directly, one calculates (a trace will be understood in the following)

$$\begin{aligned} Q\beta_z^- J_z &= J_z^- J_z^+ - [\alpha^+,\beta_z^-]^- J_z^+ + [\alpha^+,\beta_z^-]J_z - \beta_z^- D_z\alpha^+ \\ &= J_z^- J_z^+ - \beta_z^- D_z\alpha^+ \\ &\quad +\Pi_{\mathfrak{h}}[\alpha^+,\beta_z^-]\Pi_{\mathfrak{h}}[\alpha^+,\beta_z^-] + \Pi_+[\alpha^+,\beta_z^-]J_z^- \end{aligned} \tag{5.9}$$

(where one has used (5.5) in the second step), and

$$-\tfrac{1}{2}Q\beta_z^-[\alpha^+,\beta_z^-] = -J_z^-\Pi_+[\alpha^+,\beta_z^-] - \tfrac{1}{2}\Pi_{\mathfrak{h}}[\alpha^+,\beta_z^-]\Pi_{\mathfrak{h}}[\alpha^+,\beta_z^-] \ . \tag{5.10}$$

Thus, upon putting the two together and using (5.6) one finds (5.8). The same argument establishes (3.32) and hence the holomorphicity (3.31) of the twisted wave function Ψ^{TKS} we studied in section 3.4. In fact, using the holomorphic factorization of the partition function and 'pulling apart' the above argument, taking care to turn ordinary A-derivatives into covariant derivatives once they start acting on wave functions, one recovers the formulae (3.30,3.31).

5.2 The Coupling to Topological Gravity

Witten considered the coupling of the twisted $SU(2)/U(1)$ model to topological gravity in [16]. He found the coupled model by using the "Noether procedure". We will present the coupling of an arbitrary twisted Kazama-Suzuki model to topological gravity. The informal proof, above, of the metric independence of the partition function will allow us to gain extra insight to the final form of the action even though we also have arrived at the action by a step by step process.

The models that we present here represent a large new class of topological matter theories coupled to topological gravity (and hence of topological string theories).

The topological gravity multiplet is taken to be composed of the metric $\rho_{\mu\nu}$ and its superpartner $\chi_{\mu\nu}$, which transform as

$$\begin{aligned} \delta\rho_{\mu\nu} &= \chi_{\mu\nu} \\ \delta\chi_{\mu\nu} &= 0. \end{aligned} \tag{5.11}$$

Conventionally one needs to add a commuting vector C^μ, representing the reparameterization invariance. However, as argued by Witten, at the classical level such ghosts are zero and that is all that matters.

To make the writing of the action as simple as possible, we take the metric variation to be $\delta*$ which is to be understood as a variation of the complex structure $J^\nu{}_\mu$. This means, for example, that for a one form $*\omega = \omega_\nu J^\nu{}_\mu dx^\mu$, the metric variation is

$$\delta(*\omega) \equiv (\delta*)\omega \equiv \omega_\nu \delta J^\nu{}_\mu dx^\mu. \tag{5.12}$$

For a self dual one form, β, satisfying $(1+i*)\beta = 0$, there is a hidden metric dependence so that one has

$$\delta\beta = -\frac{i}{2}(\delta*)\beta \tag{5.13}$$

which is perpendicular to β, that is, $(1-i*)\delta\beta = 0$.

With our conventions the relationship between the variations (5.11) and $\delta*$ are

$$\begin{aligned} \delta J^z{}_{\overline{z}} &= -\rho^{z\overline{z}}\chi_{\overline{z}\overline{z}} \\ \delta J^{\overline{z}}{}_z &= \rho^{\overline{z}z}\chi_{zz}. \end{aligned} \tag{5.14}$$

The action which includes the coupling to topological gravity is

$$S_{TKS+TG} = S_{TKS} + \frac{1}{4\pi}\int_\Sigma \beta\delta * g^{-1}d_A g + \frac{1}{4\pi}\int_\Sigma \overline{\beta}\delta * d_A g.g^{-1}$$
$$+\frac{i}{8\pi}\int_\Sigma \delta * \overline{\beta}\delta * g\beta g^{-1} + \frac{i}{8\pi}\int_\Sigma [\overline{\beta},\overline{\alpha}]\delta * \overline{\beta} - \frac{i}{8\pi}\int_\Sigma [\beta,\alpha]\delta * \beta. \quad (5.15)$$

Apart from the last two terms this action is the same as (or, rather, the obvious generalization of) that for the $\mathbb{CP}^1$ model. The reason for the additional terms stems from the fact that for a generic coset model the anticommutator of the ghost fields in the ghost transformation rules (3.20) is not zero. For the hermitian symmetric cosets the commutators vanish and the action simplifies accordingly.

The transformation laws which leave the action invariant are

$$\begin{aligned}
\delta g &= \epsilon g\alpha + \epsilon\overline{\alpha}g \\
\delta\alpha &= -\epsilon\alpha^2 \\
\delta\overline{\alpha} &= \epsilon\overline{\alpha}^2 \\
\delta\beta &= \epsilon\Pi_-\frac{(1-i*)}{2}\left(g^{-1}d_A g - [\alpha,\beta] - \frac{i}{2}\delta * g^{-1}\overline{\beta}g\right) \\
\delta\overline{\beta} &= \epsilon\Pi_+\frac{(1+i*)}{2}\left(d_A g.g^{-1} + [\overline{\alpha},\overline{\beta}] + \frac{i}{2}\delta * g\beta g^{-1}\right) \\
\delta A &= \frac{\epsilon}{4}\Pi_{\mathfrak{h}}\left(\frac{(1-i*)}{2}i\delta * (d_A g.g^{-1} + [\overline{\alpha},\overline{\beta}]) + \frac{(1+i*)}{2}i\delta * (g^{-1}d_A g - [\alpha,\beta])\right. \\
&\quad \left. -\frac{1}{2}\delta * \delta * (g\beta g^{-1} - g^{-1}\overline{\beta}g)\right) \qquad (5.16)
\end{aligned}$$

These are somewhat unedifying, though one should notice that for the hermitian symmetric cosets these transformations coincide with those found for the $\mathbb{CP}(1)$ model in [16].

In order to facilitate the checking of the invariance of the action it is best to express S_{TKS} in the form

$$S_{TKS} = S_{G/H} - \frac{i}{2\pi}\int_\Sigma \frac{(1-i*)}{2}\beta d_A\alpha + \frac{i}{2\pi}\int_\Sigma \frac{(1+i*)}{2}\overline{\beta}d_A\overline{\alpha}. \qquad (5.17)$$

Let us compare the action and transformation rules with the considerations of the previous section. The transformation rules for g, α and $\overline{\alpha}$ are as before and require no comment. The extra term in the β_z variation is there,

as we noted before, to ensure that it remains self dual with respect to the deformed metric. The $A_{\bar{z}}$ equation of motion was required to eliminate the $J_z^{\mathfrak{h}} J_z^{\mathfrak{h}}$ part of the stress tensor (5.2). The variation of the action gives, up to fermionic terms, $\delta A_{\bar{z}} J_z^{\mathfrak{h}}$, so the appearance of $J_z^{\mathfrak{h}}$ in the transformation rule $A_{\bar{z}}$ is explained.

5.3 Preliminary Remarks on Selection Rules

At this level of generality it is difficult to make precise statements about the correspondence between this field theoretic construct and more algebro-geometric considerations. However, there are some simple observations that one can make.

1. Witten's correspondence between the field theory and algebraic geometry for the $SU(2)/U(1)$ model can be straightforwardly extended to the hermitian symmetric G/H models.

2. The fact that the $\phi - A$ system is not a natural object in cohomological theories gave rise to some complications in the analysis of [16]. This system does not arise in the case of twisted $N = 2$ G-models, i.e. when $H = \{e\}$.

3. The generalization of the superselection rule (3.39) of [16] is easily obtained. The right hand side is obtained on scaling the fields $(\chi^{zz}, \alpha) \to e^{\lambda}(\chi^{zz}, \alpha)$ and $\beta \to e^{-\lambda}\beta$. This transformation, plus a similar one for the barred fields, is an invariance of the action. The right hand side is a measure of how anomalous the transformation is.

4. The superselection rule (3.41) of [16] depends rather strongly on the choice of G and of H. On the basis of the results of [21], it can however readily be worked out for the complex Grassmannian models.

5. E.g. for the $\mathbb{CP}(n)$ models one has (see section 6.2) n bosonic operators O_i. Denoting their p'th gravitational descendant by $\tau_p(O_i)$, the generalization of the combined selection rule [16, (3.42)] for a correlator

$$\langle \prod_{a=1}^{s} \tau_{p_a}(O_i(x_a)^{r_{i,a}}) \rangle_{TKS+TG} \tag{5.18}$$

in genus g turns out to be

$$\sum_{a=1}^{s} p_a + \sum_{i=1}^{n} i \sum_{a=1}^{s} \frac{r_{i,a}}{k+n+1} = \frac{2k+3(n+1)}{k+n+1} n(g-1) + ns \ . \quad (5.19)$$

5.4 Wavefunctions (Again) and Holomorphic Factorization

We now prove that the topological models coupled to topological gravity can also be expressed as the norm of a wavefunction. This might be surprising at first, as the cross term

$$\int_\Sigma \delta * \overline{\beta} \delta * g \beta g^{-1} \quad (5.20)$$

would appear to spoil a direct factorization. Nevertheless, using the freedom afforded by the extra field B, one can show that the partition function is the norm of a wavefunction.

Indeed let

$$\Psi(A,B;*,\delta*) = \int DgD\overline{\alpha}D\overline{\beta}\,\mathrm{e}^{-kS(A,B;g;\overline{\alpha},\overline{\beta};*,\delta*)} \quad (5.21)$$

where

$$S(A,B;g;\overline{\alpha},\overline{\beta};*,\delta*) = S(A,B;g;\overline{\alpha},\overline{\beta}) - \frac{1}{4\pi}\int_\Sigma \left(d_B g . g^{-1} + \frac{1}{2}[\overline{\beta},\overline{\alpha}] \right) \delta * \overline{\beta}. \quad (5.22)$$

The dual wave function is defined by

$$\overline{\Psi(A,B;*,\delta*)} = \int DgD\alpha D\beta\,\mathrm{e}^{-kS(B,A;g;\alpha,\beta;*,\delta*)} \quad (5.23)$$

where

$$S(B,A;g;\alpha,\beta;*,\delta*) = S(B,A;g;\alpha,\beta) + \frac{1}{4\pi}\int_\Sigma \left(g^{-1} d_B g + \frac{1}{2}[\beta,\alpha] \right) \delta * \beta. \quad (5.24)$$

One now checks that

$$Z^{TG}_{TKS} = \int DB\,DA |\Psi(A,B;*,\delta*)|^2. \quad (5.25)$$

6 Observables

In the previous sections we have discussed some of the important features of the functional integral approach to the topological Kazama-Suzuki models. These observations will be used in subsequent publications to evaluate directly and explicitly correlation functions in some of these models. To that end, we will now describe the observables (physical operators).

6.1 Preliminary Remarks

Observables are local functionals of the fields (or perhaps integrals thereof) which are invariant under the symmetries of the theory. In the present context of topological G/H Kazama-Suzuki models this amounts to H-gauge invariance and invariance under the BRST-like supersymmetry δ. In fact, the δ-invariance of the action implies by standard arguments that δ-exact operators decouple so that one is actually interested in δ-cohomology classes of H-invariant operators.

To some extent, the structure of the observables in these models could be deduced from the literature on $N = 2$ superconformal field theories, e.g. [4, 11]. In these works, the chiral ring of Kazama-Suzuki models (which becomes the ring of observables of the twisted model) has been described in terms of the Lie algebra cohomology of affine algebras. One of the simplifying features of the present action-based functional integral approach is the possibility to characterize the observables directly in terms of the finite-dimensional Lie group G and its Lie algebra, loop groups or their Lie algebras never appearing explicitly.[3]

In the following, we will distinguish two classes of operators: purely bosonic operators and those depending also on the Grassmann odd fields. The latter are required for non-zero correlation functions whenever there are fermionic zero modes. Operators soaking up β and $\bar{\beta}$ zero modes can for instance be constructed from the bosonic operators using an analogue of the standard descent-procedure of topological field theories.

Operators involving α's and $\overline{\alpha}$'s, on the other hand, have a rather different flavour to them (among other things because their zero modes can be inter-

[3]This is analogous to the path integral derivation of the Verlinde formula, deeply rooted in the representation theory of loop groups, from the G/G model [24] using only some group theory of finite-dimensional groups.

preted rather directly as tangents to the (left or right) coset space G/H). We will only provide a brief description of these operators here and discuss them in detail in [21].

Turning therefore to bosonic observables, we will look for BRST and gauge invariant functionals of the group valued field g. We are deliberately ignoring a possible dependence of these operators on the connection A as we have seen before that one needs to use the A equations of motion (Schwinger-Dyson equations) to establish the metric independence of correlation functions. We are thus interested in conjugation invariant functionals $O(g)$ of g invariant under the supersymmetry

$$\delta g = g\alpha + \overline{\alpha} g \quad . \tag{6.1}$$

A rough argument suggests that in all the KS models with $\mathrm{rk}\, G = \mathrm{rk}\, H = r$ one will find r linearly independent bosonic operators, as there will be $\dim G - \dim H$ conditions imposed by δ-invariance and a $\dim G - (\dim H - r)$ dimensional space of H-invariants. Below, we will determine these functionals for the hermitian-symmetric models based on complex Grassmannian manifolds, and then we will discuss the other prototypical example of full flag manifolds G/T for G a classical group. Other models of interest could be dealt with along the same lines.

6.2 Bosonic Observables for Grassmannian Models

We will now look for observables in the Kazama-Suzuki model based on the complex Grassmannian $G(m, m+n)$ of complex m-planes in $\mathbb{C}^{m+n}$. This space can be described as the coset

$$\begin{aligned} G/H &= U(m+n)/U(m) \times U(n) = SU(m+n)/S(U(m) \times U(n)) \\ &\approx SU(m+n)/SU(m) \times SU(n) \times U(1) \quad . \end{aligned} \tag{6.2}$$

More explicitly, one embeds H into $G = SU(m+n)$ in block-diagonal form,

$$h = \mathrm{diag}(h^{(m)}, h^{(n)}) \quad , \tag{6.3}$$

where $h^{(m)}$ and $h^{(n)}$ are $U(m)$ and $U(n)$ matrices respectively, satisfying

$$\det h^{(m)} \det h^{(n)} = 1 \quad . \tag{6.4}$$

This amounts to the specification that the single $U(1)$-factor of H, which plays a special role in the hermitian-symmetric Kazama-Suzuki models, is

generated by the element $\mathrm{diag}(nI_m, -mI_n)$ (proportional to the Weyl vector $\rho_{G/H}$ of G/H).

Consequently, the Grassmann-odd scalars α and $\overline{\alpha}$ have components

$$\alpha = (\alpha_{ij}) \ , \qquad \overline{\alpha} = (\overline{\alpha}_{ji}) \ , \qquad i = 1, \ldots, m \ , \ j = m+1, \ldots, m+n \ . \quad (6.5)$$

It follows that (with the notation that indices i_k, j_k have the same range as the indices i, j above)

$$\delta g_{i_1 i_2} = 0 \ , \quad (6.6)$$

i.e. that the upper left-hand $U(m)$ block $g^{(m)}$ of g is δ-invariant,

$$\delta g^{(m)} = 0 \ . \quad (6.7)$$

The symmetry between m and n is reflected in the fact that there is also a $U(n)$ block of invariants. Namely. it follows from

$$\delta g^{-1} = -g^{-1}\delta g g^{-1} = -(\alpha g^{-1} + g^{-1}\overline{\alpha}) \ , \quad (6.8)$$

that

$$\delta(g^{-1})_{j_1 j_2} = 0 \ , \quad (6.9)$$

which we will also write as

$$\delta(g^{-1})^{(n)} = 0 \ . \quad (6.10)$$

It remains to impose gauge invariance. H gauge transformations act on $g^{(m)}$ and $(g^{-1})^{(n)}$ by conjugation with $h^{(m)}$ and $h^{(n)}$ respectively. Hence, a complete set of gauge and BRST invariant operators can be obtained as traces of $g^{(m)}$ and $(g^{-1})^{(n)}$. One possibility is to consider the traces

$$\mathrm{Tr}(g^{(m)})^l \ , \qquad l = 1, \ldots . m \quad (6.11)$$

(and likewise for $(g^{-1})^{(n)}$). However, for the cohomological interpretation of the operators it turns out to be more convenient to consider as the basic set of operators the traces of $g^{(m)}$ and $(g^{-1})^{(n)}$ in the exterior powers of the fundamental representations of $U(m)$ and $U(n)$ respectively. We thus define

$$O_l(g) := \mathrm{Tr}_{\wedge^l} g^{(m)} \ , \qquad l = 1, \ldots, m \ , \quad (6.12)$$
$$\bar{O}_l(g) := \mathrm{Tr}_{\wedge^l} (g^{-1})^{(n)} \ , \qquad l = 1, \ldots, n \ . \quad (6.13)$$

Since $\det g = 1$, there is one relation between these operators, namely

$$\det g^{(m)} \equiv O_m = \det(g^{-1})^{(n)} \equiv \bar{O}_n \ . \quad (6.14)$$

Altogether, one therefore has $\mathrm{rk}\, SU(m+n) = m+n-1$ independent basic gauge and BRST invariant operators generating the ring of observables of the topological Kazama-Suzuki model, i.e. the chiral-chiral primary ring of the $G(m, m+n)$ Kazama-Suzuki model. The ring structure (which will also depend in a subtle way on the level k) can be determined from the correlation functions of these operators. This will be done in [21] where we will show among other things that the chiral ring of the $\mathbb{CP}(m) = G(m, m+1)$ model at level k is the classical cohomology ring of the Grassmannian $G(m, m+k)$.

The functional form of the observables and the calculation of correlation functions is greatly simplified by the result of section 4 that the topological Kazama-Suzuki model can be localized and abelianized to a (perturbed) T/T model. In particular, therefore, all we ever need to know are the operators O_l and $\bar{O}_l$ evaluated for diagonal matrices $t = \mathrm{diag}(t_1, \dots, t_{n+m})$. In that case, the operators reduce to the elementary symmetric functions of the t_k,

$$O_l(t) = \sum_{1 \le i_1 < \dots < i_l \le m} t_{i_1} \dots t_{i_l} \quad , \tag{6.15}$$

$$\bar{O}_l(t) = \sum_{m+1 \le j_1 < \dots < j_l \le m+n} (t_{j_1} \dots t_{j_l})^{-1} \quad . \tag{6.16}$$

In applications we will find it convenient to parametrize the torus valued field t as

$$t = \exp i\Phi \quad , \tag{6.17}$$

where

$$\Phi = \sum_{i=1}^{n+m-1} \alpha^i \phi_i \quad , \tag{6.18}$$

the α^i being simple roots of $SU(n+m)$,

$$\alpha^i = E_{i,i} - E_{i+1,i+1} \quad , \qquad (E_{i,j})_{kl} = \delta_{ik}\delta_{jl} \quad . \tag{6.19}$$

The correspondence with the above is then

$$\begin{aligned} t_1 &= \mathrm{e}^{i\phi_1} \\ t_k &= \mathrm{e}^{i(\phi_k - \phi_{k-1})} \quad , \qquad k = 2, \dots, m+n-1 \\ t_{m+n} &= \mathrm{e}^{-i\phi_{m+n-1}} \quad . \end{aligned} \tag{6.20}$$

Let us consider some examples. In the $\mathbb{CP}(1)$ model there is one and only one scalar operator, namely

$$O_1 = g_{11} \quad . \tag{6.21}$$

In the localized theory this becomes

$$O_1 = \mathrm{e}^{i\phi_1} \quad . \tag{6.22}$$

In the $\mathbb{CP}(2)$ model, there are two scalar operators, namely

$$\begin{aligned} O_1 &= g_{11} + g_{22} \to \mathrm{e}^{i\phi_1} + \mathrm{e}^{i(\phi_2 - \phi_1)} \\ O_2 &= g_{11}g_{22} - g_{12}g_{21} \to \mathrm{e}^{i\phi_2} \quad . \end{aligned} \tag{6.23}$$

And quite generally one finds that in the $G(m, m+n)$ models the operator $O_m = \det g^{(m)}$ reduces to

$$O_m = \bar{O}_n \to \mathrm{e}^{i\phi_m} \quad . \tag{6.24}$$

As a final example, we consider the simplest Grassmannian which is not a projective space, namely $G(2,4)$. In that case one has three operators. In the localized and abelianized theory they are

$$\begin{aligned} O_1 &= \mathrm{e}^{i\phi_1} + \mathrm{e}^{i(\phi_2 - \phi_1)} \\ O_2 &= \mathrm{e}^{i\phi_2} \\ \bar{O}_1 &= \mathrm{e}^{i\phi_3} + \mathrm{e}^{i(\phi_2 - \phi_3)} \quad . \end{aligned} \tag{6.25}$$

6.3 Bosonic Observables for Classical Flag Manifolds G/T

We will now discuss the bosonic invariants for the Kazama-Suzuki models based on the Kählerian flag manifolds G/T where G is a classical group and T a maximal torus of G. The strategy is exactly as above, i.e. we are looking for functionals of g which are invariant under the supersymmetry (6.1) and the gauge transformation $g \to t^{-1}gt$, where $\alpha(\overline{\alpha})$ now takes values in the positive (negative) root spaces. In the following we shall sometimes loosely use the term root to refer also to the Lie algebra element associated with the root. A general reference for the (rudiments of) Lie algebra theory used in the following is [42].

1. $SU(n)/T$

Recall that $\dim SU(n) = n^2 - 1$ and $\mathrm{rk}\, SU(n) = n - 1$, $T = U(1)^{n-1}$, so that, on the basis of the rough argument sketched in section 6.1, we expect to find a total of $n-1$ bosonic invariants. We will see that this expectation is indeed borne out.

Let us dispose of gauge invariance first. The group valued field g is represented by an $n \times n$ matrix in $SU(n)$. Consider now the $k \times k$ block matrix

$$g_{ij}^{(k)} = g_{ij}, (1 \leq i \leq k, 1 \leq j \leq k, 1 \leq k \leq n)\,. \tag{6.26}$$

One can now easily show that $\det g_{ij}^{(k)}$ is gauge invariant as follows. For $SU(n)$, h is a diagonal $n \times n$ matrix

$$h = \mathrm{diag}(z_1, \ldots, z_n) \tag{6.27}$$

with $z_i = e^{i\theta_i}$, such that $z_1 \ldots z_n = 1$.

One sees that under $g \to h^{-1}gh$ the matrix $g^{(k)}$ transforms as

$$g^{(k)} \to (h^{-1})^{(k)} g^{(k)} h^{(k)} \quad , \tag{6.28}$$

where $h^{(k)}$ and $(h^{-1})^{(k)}$ are the upper left hand $k \times k$ blocks of h and h^{-1}, respectively. Also because of the diagonal nature of h it is easy to see that $(h^{-1})^{(k)} = (h^{(k)})^{-1}$ and hence that $\det g^{(k)}$ is gauge invariant (as are other class functions of $g^{(k)}$, of course, but these will not lead to BRST invariants).

Now let us check the first part of the supersymmetry transformation (6.1), namely

$$\delta g_{ij} = g_{im}\alpha_{mj}\,. \tag{6.29}$$

For $SU(n)$ the positive roots can be taken to be strictly upper triangular, i.e. $\alpha_{mj} = 0$ for $m \geq j$. A natural basis for the positive root spaces is

$$\alpha^{[ij]} = E_{i,j} \;\; ,, \qquad 1 \leq i < j \leq n \;\; , \tag{6.30}$$

with $(E_{i,j})_{kl} = \delta_{ik}\delta_{jl}$, associated with the positive root $\alpha^i + \ldots \alpha^{j-1}$, where α^i are the simple roots, $\alpha^i = E_{i,i} - E_{i+1.i+1}$. (Similarly the matrices belonging to the negative roots $\bar{\alpha}$ are strictly lower triangular matrices.) We note that for the elements of the submatrix $g^{(k)}$ the supersymmetry transformation is restricted to:

$$\delta g_{ij}^{(k)} = g_{im}\alpha_{mj}, 1 \leq i \leq k, 1 \leq j \leq k\,. \tag{6.31}$$

Since α is strictly upper triangular the only contributions to this sum will come for $m < j$. Hence the elements of g that appear in the sum live only in the submatrix $g^{(k)}$ itself. Therefore one may as well restrict ones attention to the part of α living in the $k \times k$ block $\alpha^{(k)}$,

$$\delta g_{ij}^{(k)} = g_{im}^{(k)}\alpha_{mj}^{(k)} \quad . \tag{6.32}$$

Then

$$\delta \det g^{(k)} = \det g^{(k)} Tr \left((g^{(k)})^{-1} \delta g^{(k)} \right) \tag{6.33}$$

$$= \det g^{(k)} Tr \alpha^{(k)} = 0 \tag{6.34}$$

since $Tr\alpha^{(k)} = 0$, $\alpha^{(k)}$ being strictly upper triangular. The same argument applies to the other half of the supersymmetry transformation $\delta g = \bar{\alpha} g$ where now the transformation of the matrices $g^{(k)}$ is affected by multiplying on the left by strictly lower triangular matrices of the same dimension. Therefore all the $\det g^{(k)},\ k = 1, \dots, n$, are gauge invariant and supersymmetric. This gives rise to $n-1$ observables as one observable, $\det g^{(n)} = \det g = 1$, is trivial. Therefore the number of scalar observables is equal to the rank of $SU(n)$.

Using the parametrization (6.17,6.18) of the torus fields, these observables take on the simple form

$$\det g^{(k)} = \mathrm{e}^{\, i\phi_k} \tag{6.35}$$

in the localized T/T theory.

2. $SP(n)/T$

First some facts about $SP(n)$. These are $2n \times 2n$ matrices A satisfying $A^T\, J\, A = J$, where J is the matrix

$$J = \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix}, \tag{6.36}$$

with I the identity matrix in $GL(n, C)$. The Lie algebra of $SP(n)$ consists of complex $2n \times 2n$ matrices

$$\mathcal{A} = \begin{bmatrix} U & W \\ V & -U^T \end{bmatrix} \tag{6.37}$$

where $V^T = V$ and $W^T = W$.

The dimension of the group is $n(2n+1)$ and the rank is n, so that the number of roots is $2n^2$. The torus is represented by diagonal $2n \times 2n$ matrices

$$h = \mathrm{diag}(z_1, \dots, z_n, \bar{z}_1, \dots, \bar{z}_n)\,, \tag{6.38}$$

with $z_i = e^{i\theta_i}$.

Again let us consider gauge invariance first. Let us write g as the $2n \times 2n$ matrix

$$g = \left[\begin{array}{cc} M & N \\ P & Q \end{array} \right] . \tag{6.39}$$

where M, N, P and Q are $n \times n$ matrices. We see that under $g \to h^{-1} g h$, the $n \times n$ matrix M transforms as

$$M^{'} = \mathrm{diag}(\bar{z}_1, \ldots, \bar{z}_n) M \mathrm{diag}(z_1, \ldots, z_n) \ . \tag{6.40}$$

Thus following the same argument as in the $SU(n)/T$ case we see that e.g. the determinant of every upper left hand $k \times k$ block of M, $\det M^{(k)}, k = 1, \ldots, n$, is gauge invariant.

For the supersymmetry we need to consider both the positive and negative roots. We will only consider the positive roots explicitly as the argument for the negative roots is then obvious. A basis for the n^2 elements of $\mathfrak{k}^+$ can be chosen as follows:

1. n matrices $A^{[i]}$, $1 \leq i \leq n$ defined as

$$A^{[i]} = E_{i,n+i}, 1 \leq i \leq n \, . \tag{6.41}$$

Referring to (6.37) this means that $U = V = 0$ and W is a diagonal matrix zero everywhere except for 1 at the i, i position.

2. $\frac{n(n-1)}{2}$ matrices $B^{[ij]}$ defined as

$$B^{[ij]} = E_{i,n+j} + E_{j,n+i}, (1 \leq i \leq n, 1 \leq j \leq, i < j) \, . \tag{6.42}$$

Again referring to (6.37), this means that $U = V = 0$ and W is a symmetric matrix with unit entries and zeroes along the diagonal.

3. $\frac{n(n-1)}{2}$ matrices $C^{[ij]}$ defined as

$$C^{[ij]} = E_{ij} - E_{n+j,n+i}, (1 \leq i \leq n, 1 \leq j \leq, i < j) \, . \tag{6.43}$$

In this case $V = W = 0$ and U is a strictly upper triangular matrix with unit entries.

Now we check for invariance under the supersymmetry transformation $\delta g = g\alpha$ with α having values in the matrices defined above. First of all we see immediately that for α in the $A^{[i]}$ and $B^{[ij]}$ direction we have $\delta M = 0$. For $\delta g = g C^{[ij]}$ we see that either the $k \times k$ block, $M^{(k)}$, is directly put to zero or

it is multiplied by a $k \times k$ matrix with entries only in the upper right hand corner and, therefore, by the same arguments that we used in the $SU(n)$ case we have again $\det M^{(k)} = 0,\ k = 1, \ldots, n$. Similar arguments apply to the other half of the supersymmetry involving the negative roots. Thus again we come to the conclusion that the number of observables is equal to the rank, in this case n.

In this case, when we localize to the T/T model, the torus valued field, $g = t$, is parametrized as

$$t = \mathrm{diag}(\mathrm{e}^{i\phi_1}, \ldots, \mathrm{e}^{i\phi_n}, \mathrm{e}^{-i\phi_1}, \ldots, \mathrm{e}^{-i\phi_n}) \tag{6.44}$$

leading to the simple form of the observable

$$\det M^{(k)} = \mathrm{e}^{i(\phi_1 + \ldots + \phi_k)}\,. \tag{6.45}$$

3. $SO(2n)/T$

We have to do this example in a slightly unusual way to make use of what we have learned from the previous examples. We would like to find the torus and the Cartan subalgebra as diagonal matrices. $SO(2n)$ is defined as $2n \times 2n$ matrices which satisfy

$$A^T A = I \tag{6.46}$$

where I is the unit matrix. The dimension of the group is $n(2n-1)$ while the rank is n and hence the number of roots are $2n(n-1)$. Let us now perform a unitary transformation on A to yield an equivalent group of matrices obeying a modified condition. Let us write

$$A = UBU^\dagger \tag{6.47}$$

so that

$$A^T A = U^{\dagger T} B^T U^T U B U^\dagger = I\,. \tag{6.48}$$

Put $K = U^T U$ to arrive at $B^T K B = K$. We will now work with this group of matrices (it is easy to check that $A \to B$ is a group homomorphism). Writing $B \simeq I + \mathcal{B}$ we see that the Lie algebra matrices $\mathcal{B}$ satisfy

$$\mathcal{B}^T K + K\mathcal{B} = 0\,. \tag{6.49}$$

A convenient choice for the matrix U is

$$U = \frac{1}{\sqrt{2}} \begin{bmatrix} i_n & -i_n \\ -I_n & -I_n \end{bmatrix}, \tag{6.50}$$

where i_n is $i \times I_n$ with I_n the identity $n \times n$ matrix. This leads to

$$K = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & I_n \\ I_n & 0 \end{bmatrix}, \tag{6.51}$$

Writing $\mathcal{B}$ in terms of $n \times n$ matrices,

$$\mathcal{B} = \begin{bmatrix} \mathcal{B}_1 & \mathcal{B}_2 \\ \mathcal{B}_3 & \mathcal{B}_4 \end{bmatrix}, \tag{6.52}$$

the condition (6.49) gives

$$\mathcal{B}_1 = -\mathcal{B}_4^T,\ \mathcal{B}_2 = -\mathcal{B}_2^T,\ \mathcal{B}_3 = -\mathcal{B}_3^T\ . \tag{6.53}$$

Here the torus is given by exactly the same matrices as in the case of $SP(n)$, i.e. eq. (6.38). A basis for the $n(n-1)$ matrices of $\mathfrak{k}^+$ belonging to the positive roots can be chosen as follows:

1. $\frac{n(n-1)}{2}$ matrices $B^{[ij]}$ defined as

$$B^{[ij]} = E_{i,n+j} - E_{j,n+i}, (1 \leq i \leq n, 1 \leq j \leq, i < j)\,, \tag{6.54}$$

that is $\mathcal{B}_1 = \mathcal{B}_3 = \mathcal{B}_4 = 0$ with $\mathcal{B}_2$ an antisymmetric matrix with unit entries.

2. $\frac{n(n-1)}{2}$ matrices $C^{[ij]}$ defined as

$$C^{[ij]} = E_{ij} - E_{n+j,n+i}, (1 \leq i \leq n, 1 \leq j \leq, i < j)\,. \tag{6.55}$$

These are the same matrices as the C's in $SP(n)$.

Now the rest of the argument follows exactly as in the $SP(n)/T$ case. There are exactly n observables (the rank of $SO(2n)$) given by $\det B^{(k)}, (k = 1, \ldots, n)$ where $B^{(k)}$ is the upper left $k \times k$ block of the group matrix B.

Upon localization, the torus valued field is parametrized in exactly the same way, eq.(6.44), as for the $SP(n)$ case and the form of the observables here is the same as in that case, eq.(6.45).

4. $SO(2n+1)/T$

The dimension of the group $SO(2n+1)$ is $n(2n+1)$ while the rank is the same as for $SO(2n)$, that is n. Hence the number of roots is $2n^2$. For the odd dimensional orthogonal group we follow the same procedure (6.47-6.49) as for the even dimensional case except that we set

$$U = \frac{1}{\sqrt{2}} \begin{bmatrix} i_n & -i_n & 0 \\ -I_n & -I_n & 0 \\ 0 & 0 & \sqrt{2} \end{bmatrix}, \tag{6.56}$$

so that

$$K = \begin{bmatrix} 0_n & I_n & 0 \\ I_n & 0_n & 0 \\ 0 & 0 & 1 \end{bmatrix} . \tag{6.57}$$

where as usual the subscript n denotes a $n \times n$ matrix. The corresponding $\mathcal{B}$ matrix can be written as

$$\mathcal{B} = \begin{bmatrix} \mathcal{B}_1 & \mathcal{B}_2 & c_1 \\ \mathcal{B}_3 & \mathcal{B}_4 & c_2 \\ d_1 & d_2 & b_1 \end{bmatrix} . \tag{6.58}$$

For the $2n \times 2n$ parts of the matrix, the conditions are the same (6.53) as for the even dimensional algebra. The constraints on the new matrices are

$$b_1 = 0, \quad c_1 = -d_2^T, \quad c_2 = -d_1^T . \tag{6.59}$$

The matrix for the torus h is the same as for $SO(2n)$ except for a 1 added at the $2n+1, 2n+1$ diagonal position,

$$h = \mathrm{diag}(z_1, \ldots, z_n, \bar{z}_1, \ldots, \bar{z}_n, 1) , \tag{6.60}$$

For the basis of the n^2 Lie algebra elements belonging to the positive roots we first choose $n(n-1)$ of them the same, $B^{[ij]}$ and $C^{[ij]}$, as in the the $SO(2n)$ case except for a last column and row of zeros added. The extra n matrices of the basis for the new positive roots are chosen as

$$D^{[i]} = e_{i,2n+1} - e_{2n+1,n+i}, \ 1 \leq i \leq n . \tag{6.61}$$

This means that $\mathcal{B}_i = 0$ and $d_1 = c_2 = 0$. This choice is important because it ensures that $\delta B^{(n)} = 0$ when α has values in the direction of these roots, where as in the $SO(2n)$ case $B^{(n)}$ is the upper left $n \times n$ block of $B \in SO(2n+1)$. Therefore with this choice of basis it follows almost immediately that, as in the $SO(2n)$ case, we have again exactly n observables given by $\det B^{(k)}$, $k = 1, \ldots, n$. Again, the localized observables have the same form as for the $SP(n)$ and $SO(2n)$ cases.

6.4 Fermionic Operators: Preliminary Remarks

If there are fermionic zero modes, then in order to have non-vanishing correlation functions they have to be soaked up (in a diffeomorphism invariant

way) by operators containing these fermionic zero modes. Typically, in a cohomological field theory such operators can be constructed from BRST invariant bosonic operators by what is known as 'descent equations'. Formally, they are a consequence of the BRST-exactness of the energy momentum tensor (see e.g. [43]). In the present case (where we are dealing with a topological field theory which is not quite of the cohomological type) there are a few subtle differences. And while this is only a minor issue and we don't want to overemphasize the differences, we will nevertheless digress briefly to review the construction.

Before doing this, we want to point out that there is also an alternative procedure for soaking up these zero modes which bypasses the complications which arise when using the descent equations and which essentially amounts to 'dropping' the zero modes in a well-defined way. First of all, we observe that as the β-zero modes β^0 do not appear in the action, one may define their BRST transformations in any way one likes without spoiling the BRST symmetry of the action. A natural choice is

$$\delta\beta^0 = \delta\overline{\beta}^0 = 0 \ . \tag{6.62}$$

With this definition, gauge invariant combinations of the β and $\bar{\beta}$ zero modes like

$$B^l = \mathrm{Tr}(\beta^0\overline{\beta}^0)^l \tag{6.63}$$

are well-defined physical operators that can be used to eliminate the fermionic zero modes from the picture without changing anything else. In particular, if one adopts this procedure one finds that there is a straightforward correspondence between correlation functions in different topological sectors. This has been discussed in terms of spectral flow in [18].

Let us now review the descent equations. The first of these expresses the fact that, given a BRST invariant scalar operator $O^{(0)}$, one can find a one-form valued operator $O^{(1)}$ satisfying

$$dO^{(0)} = \delta O^{(1)} \ . \tag{6.64}$$

The main consequence of (6.64) is that correlation functions of scalar observables are independent of the points at which they have been inserted - the hallmark of a topological field theory. What is more important for us in the present context is that (6.64) also implies that the integral of $O^{(1)}$ over a closed cycle C is also a BRST invariant operator,

$$\delta \oint_C O^{(1)} = \oint_C dO^{(0)} = 0 \ . \tag{6.65}$$

The fact that the BRST cohomology class of this new operator only depends on the homology class $[C]$ of this cycle is a consequence of the second descent equation which says that one can find a two-form valued operator $O^{(2)}$ satisfying

$$dO^{(1)} = \delta O^{(2)} \quad . \tag{6.66}$$

By the same argument as in (6.65), (6.66) implies that the integral of $O^{(2)}$ over a closed two-cycle is a new BRST invariant operator. In two dimensions the descent equations end here and one obtains the one additional operator $\int_\Sigma O^{(2)}$.

In principle, therefore, one could now take any one of the bosonic operators we constructed above and try to solve the equations (6.64) and (6.66) by hand. In practice, however, this is rather cumbersome and it is helpful to have a rather more conceptual understanding of why the descent equations hold. Our starting point is the fundamental equation of a topological field theory expressing the fact that the energy-momentum tensor is BRST-exact,

$$T_{\mu\nu} = \delta G_{\mu\nu} \quad . \tag{6.67}$$

Integrating this over a space-like hypersurface, one obtains an equation for the momentum P_μ which in operator language this can be expressed as

$$d = \delta G \tag{6.68}$$

for some fermionic operator G. Applying this to e.g. a scalar operator $O^{(0)}$ one recovers (6.64). The additional information one obtains is that the one-form operator $O^{(1)}$ can be obtained from the scalar operator by acting with the 'vector supersymmetry' G,

$$O^{(1)} = GO^{(0)} \quad . \tag{6.69}$$

Likewise one finds

$$O^{(2)} = GO^{(1)} \quad , \tag{6.70}$$

etc. In a topological conformal field theory, (6.68) can be written more explicitly as

$$\partial_z = QG_z \quad , \qquad \bar{\partial}_{\bar{z}} = \overline{Q}\bar{G}_{\bar{z}} \quad , \tag{6.71}$$

where Q ($\overline{Q}$) and G_z ($\bar{G}_{\bar{z}}$) are the left (right) moving BRST charges and supercurrents respectively.

6.5 Fermionic Descendants for Grassmannian Models

Let us now apply these considerations to Kazama-Suzuki models and, more specifically, to the observables of the Grassmannian models found above. The first thing to note is that, as we have seen in section 5.1, in the chosen Lagrangian realization of the topological Kazama-Suzuki model (as a supersymmetric gauged WZW model) the energy-momentum tensor is only BRST-exact modulo the gauge field equations of motion. This implies that a) operators depending on the gauge field (like Wilson loops) will in general not lead to topological correlation functions, and b) that we should expect to be able to solve the descent equations only modulo the equations of motions of the gauge fields.

Furthermore, the left- and right-moving BRST operators Q and $\overline{Q}$ only commute up to the fermionic equations of motion. As a consequence we will find that the descent-procedure described above, while still a useful guideline, works somewhat differently in the present context, in a sense reducing to the above only 'on shell'. This complicates the explicit determination of the one- and two-form operators but is no fundamental obstacle to arriving at fermionic descendants which have all the right properties inside correlation functions.

With this in mind, let us no determine the relevant components of the vector supersymmetry operator G. As we have seen, modulo the gauge field equations of motion, the zz-component of the energy-momentum tensor is Q-exact,

$$T_{zz} = -Q\,\mathrm{Tr}\,\beta_z^- J_z^+ \quad . \tag{6.72}$$

Choosing $\bar{z}$ as the time-coordinate in the left-moving sector, the kinetic term of the action proportional to $D_z g^{-1} D_{\bar{z}} g$ implies that, as an operator, $D_z g^{-1}$ should be represented as

$$D_z(g^{-1})_{kl} \to \frac{\delta}{\delta g_{lk}} \quad . \tag{6.73}$$

Therefore, the counterpart of the operator G_z in (6.71) is

$$G_z = \int dz\ (g\beta_z)_{li} \frac{\delta}{\delta g_{li}} \equiv \mathrm{Tr}\, g\beta \frac{\delta}{\delta g} \quad . \tag{6.74}$$

Likewise for $\bar{G}_{\bar{z}}$ one finds

$$\bar{G}_{\bar{z}} = \int d\bar{z}\ \mathrm{Tr}(\overline{\beta} g) \frac{\delta}{\delta g} \quad . \tag{6.75}$$

From these equations one can read off directly the candidates for the one-form descendants of the observables O_l of the Grassmannian Kazama-Suzuki model,

$$O_l^{(1)} = (G_z dz + \bar{G}_{\bar{z}} d\bar{z}) O_l \quad . \tag{6.76}$$

It can be verified that these indeed satisfy (6.64) and (6.65) if one uses the A-equations of motion.

For a number of reasons, however, these one-form operators appear not to be of particular interest. For one, as there are always an equal number of β and $\overline{\beta}$ zero modes, the two-form operators $\sim \beta\overline{\beta}$ are sufficient to soak these up. Moreover, the chiral ring is determined by genus zero correlation functions and in genus zero there are no non-trivial one-form operators. Finally, it seems to be a general feature of these models that the one-form operators are trivial inside correlation functions as one is localizing onto BRST-invariant configurations. In any case we will not consider further these one-form operators here. The two-form operator, on the other hand, is of interest, and it can be obtained from the scalar operator by operating on it with the commutator of G and $\bar{G}$,

$$\begin{aligned} O_l^{(2)} \equiv B_l &= \tfrac{1}{2}[\bar{G}, G]_- O_l(g) \\ &= \left[(\overline{\beta} g)_{i_1 k} (g\beta)_{l i_2} \frac{\delta}{\delta g_{i_1 k}} \frac{\delta}{\delta g_{l i_2}} + (\overline{\beta} g \beta)_{i_1 i_2} \frac{\delta}{\delta g_{i_1 i_2}} \right] O_l(g) \end{aligned} \tag{6.77}$$

This time, when verifying (6.66) one needs to use both the A equations of motion and $D_{\bar{z}}\beta_z = D_z\overline{\beta}_z = 0$ which means that one is essentially inserting only the zero modes of β and $\bar{\beta}$. In the path integral this is fine as long as none of the operators one is considering depends on the Grassmann-odd scalars α and $\overline{\alpha}$, because then integration over them will impose these equations.

As it stands, (6.77) is valid quite generally and, in particular, also for the observables of the G/T models we discussed in section 6.3 (with the understanding that β then has components β_{ji} with $i < j$ etc.).

We will of course ultimately be interested in doing calculations in the localized and abelianized theory directly (having kept only the zero modes of the fermionic fields). As a preliminary step one has to make sure that the operators we have constructed above are also good operators in the perturbed t-dependent theory we employed to establish localization. The point is that the fermionic operators we discussed above have all the requisite properties of an observable only modulo equations of motion and that these equations

of motion are t-dependent in the perturbed theory. This issue as well as the calculation and (cohomological) interpretation of correlation functions of these operators will be addressed elsewhere.

6.6 A Comment on α-Observables

In the hermitian symmetric models, observables depending on the Grassmann odd scalars α and $\overline{\alpha}$ are easy to come by and permit one to consider correlation functions in the topological sectors of the $U(1)$ gauge field in which there are zero modes of these fields. The reason why this is particularly straightforward in the hermitian symmetric models is that here α and $\overline{\alpha}$ are already BRST invariant, $\delta\alpha = \delta\overline{\alpha} = 0$. To obtain gauge and BRST invariant observables, it therefore suffices to take the invariants (traces) of the matrices $\alpha\overline{\alpha}$ and $\overline{\alpha}\alpha$. We will show in [21] that the genus zero ring structure of these observables in the hermitian symmetric G/H level k models is the classical cohomology ring of G/H - remarkably for all values of the level k.

7 Non-Trivial Bundles

In the preceding sections we have taken the point of view, even though the subgroup $F = H_L \times H_R$ of $G_L \times G_R$ is not necessarily simply connected and the bundles in question may be non-trivial, that results derived from the the WZW action (2.6) are nevertheless correct. The idea is that given some reasonable definition of the WZW theory in general that both localization and abelianization will apply and that ultimately one will be left with the already well defined torus models of section 4. This section is devoted to presenting one possible generalized WZW coset model that fulfills our expectations.

Actually almost all coset models will require such an improved theory since the question of the non-triviality of the bundles involved has further complications. The extra difficulties arise as the action of the group is the Ad action so that the group that acts faithfully *is not* F but is rather $F' = (H_L \times H_R)/Z$ where $Z = H \cap Z(G)$. This means that even though F may be simply connected the gauge group need not be. Now G bundles over a Riemann surface are classified by $\pi_1(G)$. In particular this means that one is often dealing with non-trivial F' bundles. For example consider a diagonal gauging of $SU(2) \times SU(2)$ by $SU(2)$, then as the gauging is diagonal one is

not really dealing with a diagonal $F = SU(2) \times SU(2)$ bundle but rather a diagonal $F' = SO(3) \times SO(3)$ bundle. In the following we will only consider connected and simply connected G (more general groups can be dealt with, following [28] but that would just serve to complicate matters unnecessarily here).

The most immediate consequence of the appearance of non-trivial F' bundles P is that we are no longer dealing with maps $g : \Sigma \rightarrow G$ but rather with sections of a non-trivial bundle. At some point we will have to specify patching data to fix our bundle. The fields on the various patches are then related by "gauge transformations" on the overlaps. The metric part of the action and the fermionic terms pose no quandary as they are manifestly gauge invariant so that on the overlaps it makes no difference which representative we take. We are left with the problem of making sense of the gauged WZW term $\Gamma_\Sigma(A; g)$ and, though not gauge invariant, we would also like a definition for the action of a wavefunction, $\Gamma_\Sigma(A, B; g)$, which keeps some of the properties that are required in establishing the geometric nature of the wavefunctions. Our task, then, is to give a definition of $\Gamma_\Sigma(A, B; g)$ in the case of non-trivial $H_{R,L}$ bundles so that all the usual properties, such as the Polyakov-Wiegmann identity, hold for the improved $\Gamma_\Sigma(A, B; g)$, which is denoted by $\Gamma_{\Sigma,P}(A, B; g)$ (the P subscript stands for the bundle).

By considering Riemannn surfaces with boundary, Gawedzki [28] defines a line bundle over the loop group (on the boundary $\partial\Sigma = S^1$) with the patching data being given by the exponential of the WZW action subject to an equivalence relation. The same construction, with the addition of a gauge field was used by Hori [29] to define the coset models for non-trivial bundles. We adopt and extend these constructions in order to define the action for wavefunctions. In passing we note that the observables that we have defined in section 6 do not require refinement as they are gauge invariant and hence well defined for all bundles.

In the following subsections we will, very briskly, run through the changes that one meets.

7.1 $\Gamma_{\Sigma,P}(A, B; g)$

That the definition of $\Gamma_\Sigma(A; g)$ is inadequate when non-trivial bundles are involved can be demonstrated with the use of a simple example. We consider the $SU(2)/U(1)$ theory and let $g = e^{i\phi} \in SU(2)$ lie in the torus. Such a

configuration is, of course, very special, however, we have seen in the body of the paper that such configurations are precisely those that we would like to arrive at. For such a configuration

$$\Gamma_{\Sigma}(A;g) = -\frac{1}{2\pi}\int_{\Sigma} A d\phi \tag{7.1}$$

and the problem is manifest as the connection A (and hence the integrand) is not globally defined on Σ. A gauge invariant, and hence globally defined, functional is obtained on adding a total derivative

$$\begin{aligned}\Gamma_{\Sigma,P}(A;g) &= \int_{\Sigma}\phi dA \\ &= \Gamma_{\Sigma}(A;g) - \frac{1}{2\pi}\int_{\Sigma} d(A\phi)\end{aligned} \tag{7.2}$$

To give a general definition of $\Gamma_{\Sigma,P}(A;g)$, and not just for special configurations, requires a little more work even though some, more or less, natural generalizations present themselves. Let us eliminate one such generalization of $\Gamma_{\Sigma}(A,g)$. One can express $\Gamma_{\Sigma}(A,g)$ as

$$\Gamma_{\Sigma}(A;g) = \frac{1}{12\pi}\int_{M}\left((g^{-1}d_A g)^3 + F_A(g^{-1}d_A g + d_A g g^{-1})\right) \tag{7.3}$$

which is manifestly gauge invariant. For trivial bundles this is fine, as we can extend a trivial bundle on Σ to a trivial bundle on M. For non-trivial bundles this is not the case and will not do as a definition of $\Gamma_{\Sigma,P}(A;g)$. For example let $\Sigma = S^2$ and $M = D_3$ the unit three-disc. All bundles over D_3 are trivial (almost by definition), but by restricting ones attention to the boundary of D_3 there cannot suddenly appear a non-trivial bundle. Consequently, one cannot make sense of the right-hand side of (7.3).

To get around the problem of extending non-trivial bundles into arbitrary three manifolds one takes a different tack. Rather than dealing with sections of nontrivial bundles one gives a prescription for producing maps from the sections of the bundle in question. The WZW term for these maps is well defined, being the usual one. The bulk of the work rests in establishing that nothing depends on the choices made in producing the maps from the section. We now turn to that construction.

Decompose the Riemann surface Σ along a (homotopically trivial) circle as $\Sigma = D_0 \# \Sigma_{\infty}$. In order to define a non-trivial F' bundle it is enough to specify the overlap data with a decomposition of this type.

On Σ we have g as a section of a non-trivial F' bundle P so that, with the above decomposition,

$$g_\infty|_{\partial\Sigma_\infty} = k_L^{-1} g_0|_{\partial D_0} k_R. \tag{7.4}$$

The patching data is given by the maps $(k_L, k_R) : S^1 \to (H'_L, H'_R) \equiv (H_L, H_R)/Z$. The right hand side of (7.4) is well defined, when thought of as a map into G, as the central element factors through. Operationally this means that k_L and k_R need not be periodic in θ (the angular coordinate on the bounding circle). Rather, one requires that

$$k_L(\theta + 2\pi) = z k_L(\theta) \quad k_R(\theta + 2\pi) = z k_R(\theta) \tag{7.5}$$

for some $z \in Z$.

The aim now is to extend g_∞ to a map on Σ and g_0 to a map on the sphere $\mathbb{CP}^1$. Let $\tilde{g}_0$ be a smooth map on D_0 to G such that

$$\tilde{g}_0 \vee g_\infty : \Sigma = (D_0 \# \Sigma_\infty) \to G \tag{7.6}$$

is a smooth map. Likewise let $\tilde{g}_\infty$ be a smooth map on D_∞ to G and chosen so that

$$g_0 \vee \tilde{g}_\infty : \mathbb{CP}^1 = (D_0 \# D_\infty) \to G \tag{7.7}$$

is a smooth map. We will also need that $\tilde{g}_\infty$ be a pointed map, that is at the origin $\{0\}$ of the disc $\tilde{g}_\infty(0) = I$ (or better still one may take the field $\tilde{g}_\infty = I$ on some small open neighbourhood of $\{0\}$)[4]. Notice that on the boundary circle the extensions satisfy

$$\tilde{g}_0|_{\partial D_0} = k_L^{-1} \tilde{g}_\infty|_{\partial D_\infty} k_R. \tag{7.8}$$

The gauge fields also satisfy an overlap equation; on Σ_∞ one denotes the gauge fields by A_∞ and B_∞ while on the disc D_0 one denotes them by A_0 and B_0. On the overlap

$$\begin{aligned} A_0 &= k_L^{-1} A_\infty k_L + k_L^{-1} dk_L \\ B_0 &= k_R^{-1} B_\infty k_R + k_R^{-1} dk_R. \end{aligned} \tag{7.9}$$

[4] One knows that it is always possible to find such maps. Given a map from the boundary of the disc to G one can construct a map which is the identity at the origin of the disc in the following way. The map at the boundary gives an S^1 in G, as G is simply connected one can always find a homotopy of that S^1 to the identity element of G. Think of this homotopy as being a disc with centre the identity element and boundary the S^1. Define a map from D_∞ to G by the homotopy. This homotopy will do as an example of $\tilde{g}_\infty$

We do not extend the gauge fields. The gauged WZW action is perfectly local in the gauge fields and so one does not need to improve that part of the story in this construction.

Our definition of the gauge invariant Wess-Zumino-Witten term, generalizing $\Gamma_\Sigma(A,B;g)$ is

$$\begin{aligned}\Gamma_{\Sigma,P}(A,B;g) \;&=\; \Gamma_\Sigma(A_\infty,B_\infty;\tilde{g}_0\vee g_\infty)+\Gamma_{\mathbb{CP}^1}(A_0,B_0;g_0\vee\tilde{g}_\infty)\\ &\quad -\Gamma_{\mathbb{CP}^1}(\tilde{g}_0\vee\hat{g}_\infty)+i\mathcal{C}^T_\Sigma(a_L,b_R;\tilde{g}_\infty).\end{aligned} \tag{7.10}$$

The map $\hat{g}_\infty$ is defined by

$$\hat{g}_\infty=\tilde{g}_\infty^{\tilde{k}_\infty}=\tilde{k}^{-1}_{L\infty}\tilde{g}_\infty\tilde{k}_{R\infty} \tag{7.11}$$

and the connections a_{H_L} and b_{H_R} are

$$a_L=\tilde{k}_{L\infty}d\tilde{k}^{-1}_{L\infty}\qquad b_R=\tilde{k}_{R\infty}d\tilde{k}^{-1}_{R\infty} \tag{7.12}$$

where $\tilde{k}_{L,R\infty}$ is an extension of $k_{L,R}$ to the interior of the disc minus the origin $D_\infty-\{0\}$.

For non-trivial H bundles we are not be able to extend the $k_{(L,R)}$ to the whole disc while remaining in $H_{(L,R)}$, for if we could we would have produced a homotopy between the k and a constant map and hence be dealing with a trivial bundle[5]. Without such an extension it appears that $\hat{g}_\infty$ remains undefined, however, all is well as we have taken the extension $\tilde{g}_\infty(0)=I$, as then at this point

$$\hat{g}_\infty=k_L^{-1}k_R \tag{7.13}$$

is globally defined (the non-trivial nature of the bundle comes from either explicit $U(1)$ factors or from the modding through by Z). Finally

$$\mathcal{C}_\Sigma(A,B;g)=\Gamma_\Sigma(A,B;g)-\Gamma_\Sigma(g) \tag{7.14}$$

is a local functional. $\tilde{g}_0\vee g_\infty$ is a smooth map on Σ, $g_0\vee\tilde{g}_\infty$ is a smooth map on $\mathbb{CP}^1$ as is $\tilde{g}_0\vee\hat{g}_\infty$. Thus every term in the generalized WZW functional is well defined. When $H_L=H_R$ and $A=B$, our generalized gauged WZW term devolves to Hori's [29] definition of a generalized gauged WZW functional $\Gamma_{\Sigma,P}(A;g)$

$$\Gamma_{\Sigma,P}(A;g)=\Gamma_{\Sigma,P}(A,A;g), \tag{7.15}$$

[5]By allowing the $\tilde{k}$ fields to take values in G we can find extensions to the whole disc since $\pi_1(G)=\pi_2(G)=0$. The subsequent "trivialization" of the bundle means that the gauge fields will take values in $\mathfrak{g}$ even though one integrates only over a $\mathfrak{h}$'s worth.

with $b_H = a_H$. Let us also note that gauge transformations (h_L, h_R) satisfy

$$h_{L\,0} = k_L^{-1} h_{L\,\infty} k_L^{-1}, \quad h_{L\,0} = k_R^{-1} h_{R\,\infty} k_R^{-1} \tag{7.16}$$

on the overlap. The Z identifications factor through, so that the patching data for the gauge transformations is given completely by the (H_L, H_R) bundle.

We now list, without proof, some of the most important properties of this construction.

Properties:

1. $\Gamma_{\Sigma,P}(A, B; g)$ does not depend on the extension fields $\tilde{g}_0$, $\tilde{g}_\infty$ and $\hat{g}_\infty$.

2. $\Gamma_{\Sigma,P}(A, B; g)$ does not depend on the extension maps $\tilde{k}_L$ or $\tilde{k}_R$.

3. When the bundles are trivial $\Gamma_{\Sigma,P}$ reverts to Γ_Σ.

4. $\Gamma_{\Sigma,P}(A, B; g)$ satisfies the generalized Polyakov-Wiegmann identity
$$\Gamma_{\Sigma,P}(A^{h_L}, B; h_L^{-1} g) = \Gamma_{\Sigma,P}(A, B; g) + \Gamma_{\Sigma,P}(h_L^{-1}) + \frac{i}{4\pi}\int_\Sigma A dh_L h_L^{-1} \tag{7.17}$$
where
$$\Gamma_{\Sigma,P}(h_L^{-1}) = \Gamma_{\Sigma,P}(0; h_L^{-1}) \tag{7.18}$$
with $h_{L\,0} = k_L^{-1} h_{L\,\infty} k_L$ at the boundary. There is a similar equation for B gauge transformations.

The first two assertions tell us that $\Gamma_{\Sigma,P}$ is well defined and does not depend on any choices that we make. The third and fourth tell us that it is a "natural" generalization of Γ_Σ. The proofs of these statements are straightforward but rather tedious (they amount essentially to repeated use of the Polyakov Wiegmann identities).

7.2 Action and Wavefunctions

The action that we take to generalize to non-trivial bundles is

$$S_{\Sigma,P}(A, B; g) = -\frac{1}{8\pi}\int_\Sigma g^{-1} d_{\mathcal{A}} g * g^{-1} d_{\mathcal{A}} g - i\Gamma_{\Sigma,P}(A, B; g) \tag{7.19}$$

which, by virtue of property (4) above, satisfies the generalized Polyakov-Wiegmann identities

$$\begin{aligned} S_{\Sigma,P}(A^{h_L}, B; h_L^{-1}g) &= S_{\Sigma,P}(A,B;g) + i\Gamma_{\Sigma,P}(h_L) - \frac{i}{4\pi}\int_\Sigma A\, dh_L\, h_L^{-1} \\ S_{\Sigma,P}(A, B^{h_R}; gh_R) &= S_{\Sigma,P}(A,B;g) - i\Gamma_{\Sigma,P}(h_R) + \frac{i}{4\pi}\int_\Sigma B\, dh_R h_R^{-1} \end{aligned} \tag{7.20}$$

The difference between the gauge transformed action and the action depends neither on the metric nor on the section g. Furthermore on setting $H_L = H_R$ and taking $A = B$ we obtain a gauge invariant action which is the action that we adopt when dealing with non-trivial bundles and coincides with that introduced by Hori [29].

Given the action (7.19) we may form

$$\Psi_P(A,B;g) = e^{-kS_{\Sigma,P}(A,B;g)} \tag{7.21}$$

and wavefunctions

$$\Psi_P(A,B) = \int Dg e^{-kS_{\Sigma,P}(A,B;g)} = \int Dg \Psi_P(A,B;g) \tag{7.22}$$

where the path integral is over the space of sections of the bundle $\{P\}$, that is, over (g_0, g_∞) satisfying (7.4). When P is trivial it is possible to interpret the wavefunction as a holomorphic section of a product of line bundles $\mathcal{L}_1^{\otimes k} \otimes \mathcal{L}_2^{\otimes(-k)} \to \mathcal{A} \otimes \mathcal{B}$. However, in the present situation, it is better to think of the space of gauge fields (A,B) as one space $\mathcal{C}$ with gauge group $(H_L, H_R)/Z$. On $\mathcal{C}$ there is still a symplectic two-form given by (2.22) and one is given a single line bundle $\mathcal{L} \to \mathcal{C}$ whose curvature agrees with the symplectic two-form. One expects that $\Psi_P(A,B)$ is a holomorphic section of $\mathcal{L}^{\otimes k}$. Under a gauge transformation one finds

$$\Psi_P(A^{h_L}, B^{h_R}) = e^{ik\Phi_P}\Psi_P(A,B) \tag{7.23}$$

where

$$\Phi_P = \Gamma_{\Sigma,P}(h_R) - \Gamma_{\Sigma,P}(h_L) + \frac{1}{4\pi}\int_\Sigma \left(A dh_L h_L^{-1} - B dh_R h_R^{-1} \right) \tag{7.24}$$

The generalized Polyakov-Wiegmann identities now guarantee that we will obtain Ward identities analogous to those that hold in the case of trivial

bundles. If one takes infinitesimal gauge transformations, $h_L = I + u_L + \ldots$ and $h_R = I + u_R + \ldots$, the variation of the action (7.19) is

$$\delta S(A, B; g) = \frac{i}{4\pi} \int_\Sigma (u_L dA - u_R dB) \,. \tag{7.25}$$

The variation (7.25) is exactly as we wrote it for 'trivial' bundles (2.15). Notice that while $\Gamma_\Sigma(I + u + \ldots) = 0 + \ldots$, we have instead that $\Gamma_{\Sigma,P}(I + u_L + \ldots) = \frac{1}{4\pi} \int_\Sigma d(u_L A)$ (so that in (7.25) the exterior derivative is indeed in the correct spot).

Ψ_P satisfies

$$\begin{aligned} \left(D^A_\mu \frac{\delta}{\delta A_\mu} + \frac{ik}{4\pi} \epsilon^{\mu\nu} \partial_\mu A_\nu \right) \Psi_P(A, B) &= 0 \\ \left(D^B_\mu \frac{\delta}{\delta B_\mu} - \frac{ik}{4\pi} \epsilon^{\mu\nu} \partial_\mu B_\nu \right) \Psi_P(A, B) &= 0 \end{aligned} \tag{7.26}$$

with the variations and gauge fields appearing to be understood patch-wise. Likewise the following equations hold

$$\begin{aligned} \frac{D}{DA_{\bar z}} \Psi_P(A, B) &= 0 \\ \frac{D}{DB_z} \Psi_P(A, B) &= 0 \end{aligned} \tag{7.27}$$

providing one also understands them patch-wise. Put together (7.26) and (7.27) give the more covariant

$$\begin{aligned} \left(D^A_\mu \frac{D}{DA_\mu} + \frac{ik}{4\pi} \epsilon^{\mu\nu} F(A)_{\mu\nu} \right) \Psi_P(A, B) &= 0 \\ \left(D^B_\mu \frac{D}{DB_\mu} - \frac{ik}{4\pi} \epsilon^{\mu\nu} F(B)_{\mu\nu} \right) \Psi_P(A, B) &= 0 \end{aligned} \tag{7.28}$$

7.3 Factorization

The wavefunctions have an important convolution property when the gauge field B takes its values in $\mathfrak{g}$. Fix the Riemann surface Σ. Let g be a section of a $(H_L \times G)/Z$ bundle P_1 with connection (A, B) and let h be a section of

a $(G \times H_R)/Z$ bundle P_2 with connection (B, C) (for this to make sense we require $Z = H_R \cap Z(G) = H_L \cap Z(G)$). Then one finds

$$\int DB \Psi_{P_1}(A, B; g) \Psi_{P_2}(B, C; h) = \Psi_{P_3}(A, C; gh). \tag{7.29}$$

The integration over the gauge field B certainly makes sense as the product of the wavefunctions is invariant under G gauge transformations. Notice that the wavefunction on the right-hand side of this expression involves a section gh of a $(H_L \times H_R)/Z$ bundle P_3 with connection (A, C).

Setting $H = H'$ and $A = C$ in (7.29) we reproduce the fact that a G/H model can be expressed as a norm of wavefunctions. Consequently we are halfway to establishing holomorphic factorization.

Even when B does not live in all of $\mathfrak{g}$ the integral of a product of wavefunction makes sense (is gauge invariant) and leads to a respectable coset model. This remains true when we couple to fermions.

7.4 Metric Variation

The metric dependence of the coset models rests in $S_0(A, B; g)$ where

$$S_0(A, B; g) = -\frac{1}{8\pi} \int_\Sigma g^{-1} d_{(A,B)} g * g^{-1} d_{(A,B)} g. \tag{7.30}$$

We have not tampered with this part of the action, so that the metric variation is as for the case of trivial bundles. Furthermore, as the wavefunctions continue to satisfy (7.26) we find that a variation of the metric can be exactly cancelled by a second variation of the B field, that means that

$$\nabla^{(1,0)} \Psi_P(A, B; \rho) = 0. \tag{7.31}$$

As discussed above

$$\int_\Sigma \delta\rho_{\bar{z}\bar{z}} \operatorname{Tr} \frac{D}{DB_{\bar{z}}} \frac{D}{DB_{\bar{z}}} \tag{7.32}$$

must be understood patch-wise. The condition (7.31) is taken to mean that Ψ_P is an anti-holomorphic section of $\mathcal{W}$ for non-trivial P. Thus, from this point of view, holomorphic factorization has been established.

8 Applications to the Kazama-Suzuki Coset Models

The discussion of the previous section more or less covers the wavefunctions for the supersymmetric and topological theories as well. One keeps the fermionic part of the actions as given at the start of this paper as they make sense as they stand. In this section we will establish that replacing the bosonic part of the action $S_{\Sigma}(A, B; g)$ by $S_{\Sigma,P}(A, B; g)$ does not spoil the supersymmetry. Consequently the supersymmetric wavefunctions are taken care of.

In order to explicitly solve the topological Kazama-Suzuki models we had need of two main ingredients, localization and abelianization. Here we will show that for non-trivial bundles these techniques still apply. The reason for this is that working locally, on patches, both techniques apply just as they did for the trivial bundles, global information is regained by taking into account the patching data $(k_L, k_R) = (k, k)$.

8.1 Supersymmetry and Coupling to Topological Gravity

The supersymmetry transformations (2.43) still leave the action (with the bosonic action being S_P) invariant providing we interpret them appropriately. On varying the group element g we understand that we are varying g_0 and g_∞ on their respective patches. This makes sense as, on the right-hand side, the fermions (Grassmann variables) have the same patching properties on the overlap. Now varying g in (7.19) leaves us with a local gauge invariant expression which is the same that obtained on varying $S_{\Sigma}(A, B; g)$. Hence the supersymmetry of the various theories is guaranteed.

The same argument can be applied for the coupling to topological gravity. The metric variation of the improved WZW action agrees (by construction) with the variation of the usual WZW action. The crucial point is that the gauge field variation of $S_{\Sigma,P}(A, A; g)$ also agrees with the gauge field variation of $S_{\Sigma}(A; g)$. This is true as in (7.19) the gauge fields couple only to (g_0, g_∞). Consequently in the action for the twisted Kazama Suzuki model coupled to topological gravity one can replace $S_{\Sigma}(A, g)$ with $S_{\Sigma,P}(A; g)$ and retain the supersymmetry.

8.2 Localization

To implement localization in the case of trivial bundles we decomposed the group element g in the G/H Kazama-Suzuki models as

$$g = he^{i(\phi^+ + \phi^-)} \tag{8.1}$$

and scaled the coset fields $\phi^\pm \to t^{-1/2}\phi^\pm$. To see that this strategy is justified even for non-trivial bundles we work patch-wise. We let

$$\begin{aligned} g_0 &= h_0 e^{i(\phi_0^+ + \phi_0^-)} \\ g_\infty &= h_\infty e^{i(\phi_\infty^+ + \phi_\infty^-)} \end{aligned} \tag{8.2}$$

and on the overlap we have

$$h_0 e^{i(\phi_0^+ + \phi_0^-)} = k^{-1} h_\infty e^{i(\phi_\infty^+ + \phi_\infty^-)} k \tag{8.3}$$

In the large t limit after scaling we find that

$$h_0 \left(1 + \frac{i}{\sqrt{t}}(\phi_0^+ + \phi_0^-) + \ldots\right) = k^{-1} h_\infty \left(1 + \frac{i}{\sqrt{t}}(\phi_\infty^+ + \phi_\infty^-) + \ldots\right) k \tag{8.4}$$

which splits into two equations

$$h_0 = k^{-1} h_\infty k \tag{8.5}$$

and

$$(\phi_0^+ + \phi_0^-) = k^{-1}(\phi_\infty^+ + \phi_\infty^-) k \tag{8.6}$$

that is,

$$\phi_0^+ = k^{-1}\phi_\infty^+ k, \quad \phi_0^- = k^{-1}\phi_\infty^- k. \tag{8.7}$$

The first equation (8.5) is the statement that h is a section of a non-trivial H' bundle. The second equation (8.7) matches the conditions that are placed on the Grassmann variables. Putting the two together we learn that on localizing the ratio of determinants coming from the Grassmann variables and those from $\phi^\pm$ is, up to the standard anomaly, unity and we are ultimately left with a H/H' model for a non-trivial H' bundle.

8.3 Abelianization

We have now reduced the problem to calculating expectation values of certain operators in a H/H' model at some level. The H valued group fields (sections) are again defined locally with overlap data specified. On each patch we can, up to the usual obstruction, abelianize the group valued fields. Let the H sections be denoted by h. On Σ_∞ we have $h_\infty = l_\infty^{-1} t_\infty l_\infty$, while on D_0, $h_0 = l_0^{-1} t_0 l_0$, with $(t_0, t_\infty) \in T$. Abelianizing patch-wise means that on the overlap we have

$$t_0(\theta) = m_{0\infty}^{-1}(\theta) t_\infty(\theta) m_{0\infty}(\theta), \;\; m_{0\infty}(\theta) = l_\infty k l_0^{-1}. \tag{8.8}$$

We can organize things so that $m_{0\infty} \in T$ and, consequently, that $t_0 = t_\infty$ on the overlap. For globally defined bundles $m_{0\infty}$ would be periodic in θ, however, as k is not periodic then neither is $m_{0\infty}$. This feeds its way into the first Chern class associated with the torus bundle. Put another way, the torus field strength now encodes the information associated with the original non-triviality of the H' bundle as well as the non-triviality of the 'liberated' torus bundles.

To see this explicitly for an $SU(n)/\mathrm{Ad}SU(n)$ bundle, let the patching data be specified by

$$k_p(\theta) = e^{ip/n\theta\lambda_{n-1}}, \tag{8.9}$$

where λ_{n-1} is the fundamental weight $\frac{1}{n}(I_{n-1}, 1-n)$ (I_m is the $m \times m$ unit matrix), so that $k_p(\theta + 2\pi) = e^{2\pi ip/n} k_p(\theta)$ with $p = 0, \ldots, n-1$. On abelianization the patching data for the $U(1)^{n-1}$ gauge fields is specified by the $m_{0\infty}$. The Chern classes are then read off as

$$\begin{aligned} c_1 &= \frac{1}{2\pi i} \oint m_{0\infty}^{-1} dm_{0\infty} \\ &= \frac{p}{n}\lambda_{n-1} + \sum_{i=1}^{n-1} n^i \alpha_i \end{aligned} \tag{8.10}$$

where the n^i are integers and the α_i are simple roots. One arrives at the right-hand side in the following way. Since, l_0 and l_∞ are periodic we have

$$m_{0\infty}(\theta + 2\pi) = e^{2\pi ip/n} m_{0\infty}(\theta) \tag{8.11}$$

and consequently

$$m_{0\infty}(\theta) = k_p(\theta) e^{i\sum_{i=1}^{n-1} n^i \alpha_i \theta}. \tag{8.12}$$

The upshot is that the Chern class measures the non-triviality of the original bundle (measured by p) and the non-triviality of the liberated torus bundles (measured by the n^i).

A detailed exposition of how to deduce the Chern classes for the bundles that arise in the Grassmannian coset models is given in [21].

8.4 Selection Rules

Before gauging, the G WZW action has a large global symmetry group, including $G_L \times G_R$. Once the action is gauged, however, the global invariance is considerably reduced. Indeed gauged WZW terms appear in the bosonization of fermionic systems precisely because they capture the the non-invariance of the fermionic theory under chiral transformations. Just as in the fermionic theory, global chiral transformations in the gauged WZW theories yield selection rules.

These selection rules, that govern the vanishing of correlation functions, are easily derived at the level of the abelianized theory [16, 21]. They are manifest at the abelian level and correspond to shifts of the torus group valued fields. However, one may wonder how to arrive at them before abelianization. To derive a selection rule we consider transformations on the fields $g \rightarrow hg$ where h is constant and commutes with the gauge field A, g and with k. The metric part of the action is invariant under this change of variables and we must, therefore, consider the effect on the WZW functional. We know that

$$\Gamma_{\Sigma,P}(A;hg) = \Gamma_{\Sigma,P}(A;g) + \Gamma_{\Sigma,P}(h) \tag{8.13}$$

for such an h. One would think that as h is constant then it would be possible to choose its extension into the bounding three manifold to be constant so that $\Gamma_{\Sigma,P}(h) = 0$. This expectation is naive, as in the definition of $\Gamma_{\Sigma,P}(A;hg)$ we meet a field $\widetilde{hg}_\infty = \tilde{h}_\infty \tilde{g}_\infty$ which should satisfy $\widetilde{hg} = I$ at the origin of the disc. Now we have set $\tilde{g}_\infty = I$ at the origin so that h cannot be taken to be constant throughout. Consequently $\Gamma_{\Sigma,P}(h)$ need not be zero.

We are allowed to consider any extension of the element h which is consistent with the condition that $\tilde{h}_\infty(0) = I$ and that it is constant on Σ_∞ and D_0 (taking the same value on the two domains). On D_∞ (thought of as a unit disc) we set $\tilde{h}_\infty(r,\theta) = \tilde{h}(r)$ and demand that for some $r_0 \neq 0$,

$$\tilde{h}(r) = 0, \quad 0 \leq r \leq r_0. \tag{8.14}$$

Furthermore, we demand that $\tilde{k}_\infty(r,\theta) = \tilde{k}(\theta)$ and that the extensions $\tilde{h}$ and $\tilde{k}$ commute as matrices regardless of which points they are evaluated at. With these simplifications plugging into the definition of $\Gamma_{\Sigma,P}(h)$ we arrive at

$$S_{\Sigma,P}(A;hg) = S_{\Sigma,P}(A;g) + \frac{\mathrm{Tr}}{2\pi}\oint_{S^1} \lambda\tilde{k}^{-1}d\tilde{k}, \tag{8.15}$$

where $h = e^{i\lambda}$.

As a simple example, if we consider the "$SU(2)/U(1)$" model we note that $\oint k^{-1}dk = i\int_\Sigma dA$ so that, in this case,

$$S_{\Sigma,P}(A;hg) = S_{\Sigma,P}(A;g) + \frac{i}{2\pi}\int_\Sigma \mathrm{Tr}\,\lambda dA \tag{8.16}$$

which agrees with the standard chiral anomaly and the result used in [16].

Acknowledgements

The research of M.B. was supported by a grant from the EC within the framework of the HCM (Human Capital and Mobility) programme. We thank K.S. Narain for numerous discussions and L. Alvarez-Gaumé for an encouraging conversation. We would like to thank the referee for supplying us with some references and suggestions which led us to investigate the issues addressed in sections 7 and 8.

References

[1] Y. Kazama, H. Suzuki, Nucl. Phys. B321 (1989) 232-268; Phys. Lett. B216 (1989) 112-116.

[2] E. Witten, Nucl. Phys. B223 (1983) 422-432; Commun. Math. Phys. 92 (1984) 455-472.

[3] E. Witten, Commun. Math. Phys. 144 (1992) 189-212.

[4] W. Lerche, C. Vafa, N. Warner, Nucl. Phys. B324 (1989) 427-474.

[5] P. Goddard, A. Kent, D. Olive, Commun. Math. Phys. 103 (1986) 105-130.

[6] C. Schweigert, Commun. Math. Phys. 149 (1992) 425.

[7] P. Fendley, W. Lerche, S. Mathur, N. Warner, Nucl. Phys. B348 (1991) 66-88.

[8] W. Lerche, N. Warner, Nucl. Phys. B358 (1991) 571-599.

[9] D. Gepner, Nucl. Phys. B322 (1989) 65-81.

[10] D. Gepner, Commun. Math. Phys. 141 (1991) 381-411; Commun. Math. Phys. 142 (1991) 433-491.

[11] S. Hosono, A. Tsuchiya, Commun. Math. Phys. 136 (1991) 451-486.

[12] K. Bardacki, M.B. Halpern, Phys. Rev D3 (1971) 2493-2506. M.B. Halpern, Phys. Rev. D4 (1971) 2398-2401.

[13] K. Gawedzki, A. Kupiainen, Nucl. Phys. B320 (1989) 625-668.

[14] D. Karabali, H. Schnitzer, Nucl. Phys. B329 (1990) 649-666.

[15] H. Schnitzer, Nucl. Phys. B324 (1989) 412-426.

[16] E. Witten, Nucl. Phys. B371 (1992) 191-245.

[17] S. Nakatsu, Prog. Theor. Phys. 87 (1992) 795.

[18] T. Nakatsu, Y. Sugawara, Nucl. Phys. B405 (1993) 695-743; hep-th/9304029.

[19] M. Henningson, Nucl. Phys. B413 (1994) 73-83; hep-th/9307040.

[20] M. Henningson, Nucl. Phys. B423 (1994) 631-638; hep-th/9402122.

[21] M. Blau, F. Hussain, G. Thompson. *Grassmannian Topological Kazama-Suzuki Models and Cohomology*, ICTP and ENSLAPP preprint IC/95/341, ENSLAPP-L-557/95, 44 p., *to appear*.

[22] M. Blau, G. Thompson, J. Math. Phys. 36 (1995) 2192-2236; hep-th/9501075.

[23] M. Blau, G. Thompson, Nucl. Phys. B439 (1995) 367-394; hep-th/9407042.

[24] M. Blau, G. Thompson, Nucl. Phys. B408 (1993) 345-390; hep-th/9305010.

[25] E. Witten, *The Verlinde Algebra and the Cohomology of the Grassmannian*, hep-th/9312104.

[26] M. Blau, G. Thompson, Commun. Math. Phys. 172 (1995) 639-660; hep-th/9412056.

[27] J. Nunes, H. Schnitzer, *Field Strength Correlators for Two Dimensional Yang-Mills Theories on Riemann Surfaces*, Brandeis preprint BRX-TH-376, hep-th/9510154; *The Master Field for 2D QCD on the Sphere*, Brandeis preprint BRX-TH-385, hep-th/9510155.

[28] K. Gawedzki, *Wess-Zumino-Witten Conformal Field Theory*, Constructive Quantum Field Theory II, Erice (1988).

[29] K. Hori, *Global Aspects of Gauged Wess-Zumino-Witten Models*, hep-th/9411134.

[30] A.N. Schellekens, Nucl. Phys. B366 (1991) 27-73. J.Fuchs, A.N. Schellekens, C. Schweigert, Nucl. Phys. B461 (1996) 371-406; hep-th/9509105.

[31] W. Lerche, Nucl. Phys. B434 (1995) 445-474; hep-th/9312188.

[32] K. Gawedzki, *Constructive Conformal Field Theory*, in *Functional Integration, Geometry and Strings*, eds. Z. Hava and J. Sobczyk (Birkhauser, 1989).

[33] M. Blau, P. Degiovanni, G. Thompson, *unpublished*.

[34] J. Fuchs, C. Schweigert, Nucl. Phys. B411 (1994) 181-222; hep-th/9304133.

[35] C. Johnson, Mod. Phys. Lett. A10 (1995) 549, hep-th/9409062; P. Berglund, C. Johnson, S. Kachru, P. Zaugg, *Heterotic Coset Models and (0,2) String Vacua*, hep-th/9509170.

[36] P. Spindel, A. Sevrin, W. Troost, A. Van Proeyen, Nucl. Phys. B308 (1988) 662; Nucl. Phys. B311 (1988) 465.

[37] A. Giveon, E. Witten, Phys. Lett. B332 (1994) 44-50; hep-th/9494184.

[38] E. Witten, Commun. Math. Phys. 118 (1988) 411-449.

[39] T. Eguchi, S.-K. Yang, Mod. Phys. Lett. A5 (1990) 1693-1701.

[40] T. Eguchi, S. Hosono, S.-K. Yang, Commun. Math. Phys. 140 (1991) 159-168.

[41] E. Witten, *Mirror Manifolds and Topological Field Theory*, in *Essays on Mirror Manifolds* (ed. S.T. Yau), International Press, Hong Kong (1992); hep-th/9112056.

[42] T. Bröcker, T. tom Dieck, *Representations of Compact Lie Groups*, Springer, New York (1985).

[43] R. Dijkgraaf, E. Verlinde, H. Verlinde, Nucl. Phys. B352 (1991) 59-86.